Subjective Probabilities on a State Space

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Abstract

This paper extends the work of Karni (2009) in two distinct directions. First, it generalizes the model allowing for action-bet interaction and, consequently, the possibility that the decision-maker's risk attitudes may be affected by his choice of action. Second, it extends the analytical framework to include a state space and advances a choice-based definition of subjective probabilities that represent the beliefs of Bayesian decision makers regarding the likelihoods of events, thus resolving a fundamental difficulty with the definition of subjective probabilities.

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1 Introduction

Beginning with Ramsey (1927) and de Finetti (1937) and culminating with Savage (1954), the concept of choice-based subjective probabilities has been a subject of much interest and some controversy. With rare exceptions, subjective probabilities were treated as an aspect of the representation of a decision maker's preference relation, which *defines* his degree of belief regarding the likelihood of events. According to this approach, whether there exist, in the mind of the decision maker, beliefs that can be quantified by probabilities, is immaterial as long as his preferences may be represented using such probabilities.[1]

A radically different approach considers the decision maker's beliefs to be a cognitive phenomenon that feeds into the decision-making process. According to this approach the subjective probabilities *measure*, rather than define, the decision maker's beliefs.

A fundamental difficulty with the former approach is that the uniqueness of the subjective probabilities requires the use of some normalization, which is not implied by the structure of the preference relations and, consequently, is devoid of choice-theoretic meaning. For example, the definitions of subjective probabilities in the works of Savage (1954), Anscombe and Aumann (1963), Machina and Schmeidler (1992, 1995), and Gilboa and Schmeidler (1989),

[1] This is the approach taken by Savage (1954) and all subsequent contributions invoking Savage's analytical framework. These include the subjective expected utility model of Anscombe and Aumann (1963), the probabilistic sophisticaded choice theory of Machina and Schmeidler (1992, 1995), and the multi-prior model of Gilboa and Schmeidler (1989) .

are based on the convention that constant acts (that is, functions that assign the same consequence to every state) yield the same utility in every state. This convention is not testable within the analytical framework of these models and must be taken on faith.[2] If the consequences correspond to state-contingent levels of wealth, then the subjective probabilities are defined by the marginal rates of substitution between state-contingent payoffs at certainty.[3]

A "measurement" approach to the definition of subjective probabilities was recently explored in Karni (2009) in the context of Bayesian decision model. He dispenses with Savage's notion of a state space, proposing instead a new analytical framework that consists of a set, Θ, of effects (physical phenomena on which the decision maker may place bets and which may or may not impact his well-being); a set, A, of actions (initiatives by which the decision maker believes he can affect the likelihoods of ensuing effects); a set, B, of bets on these effects; and a set of informative and not informative signals, $\bar{X}$, received before taking actions and choosing bets. The choice set, $\mathcal{I}$, consists of information-contingent plans (strategies) for choosing actions and bets (that is, a strategy is a function $I : \bar{X} \to A \times B$). In this framework, Karni (2009) develops a complete, choice-based, Bayesian decision theory in which decision makers' preferences are represented by

$$I \mapsto \sum_{x \in \bar{X}} \left[\sum_{\theta \in \Theta} \pi \left(\theta, x \mid a_{I(x)} \right) \left[u \left(b_{I(x)} \left(\theta \right), \theta \right) + v \left(a_{I(x)} \right) \right] \right] \tag{1}$$

where $a_{I(x)}$ and $b_{I(x)}$ are the action and bet assigned to the observation x by the strategy I;

[2] In the same vain, Karni's (1993) definition of subjective probability with state-dependent preferences uses the boundedness of the utility function to obtain the required normalization. Karni and Schmeidler (1993) use the marginal rates of substitution among payoffs in different states to normalize the utility functions.

[3] See Karni Schmeidler (1993) and Nau (1995) .

$\{u(\cdot,\theta)\}_{\theta\in\Theta}$ are effect-dependent utility functions on the monetary payoffs of the bets; v is the (dis)utility of actions; and $\{\pi(\cdot,\cdot \mid a)\}_{a\in A}$ is a unique family of action-dependent, joint, subjective probabilities distributions on $\Theta\times\bar{X}$ such that the prior distributions $\{\pi(\cdot \mid o,a)\}_{a\in A}$ and the posterior distributions $\{\pi(\cdot \mid x,a)\}_{a\in A}$ on Θ are linked by Bayes rule and represent the decision maker's prior and posterior beliefs. It is worth underscoring the fact that the family of action-dependent, joint, subjective probabilities distributions $\{\pi(\cdot,\cdot \mid a)\}_{a\in A}$, is the only family of such distributions that incorporates new information solely through its effect on the decision-maker's beliefs rather than his tastes. In other words, the decision-maker's posterior preferences are obtained from the prior preferences by the updating of the subjective probabilities living the utility functions intact.

This paper builds upon and extends the work of Karni (2009). The following example lends concrete meaning to the abstract terms mentioned above and serves to motivate the extensions of this work. Consider a decision maker faced with the prospect of an approaching hurricane. The decision maker must make a plan that, contingent on the weather reports, may include boarding up his house, moving his family to a shelter, and betting on the storm's damage (that is, taking out insurance). The uncertainty is resolved once the weather forecast is obtained, the plan is put into effect, the storm passes, and its path and force have been determined.

In this example, effects correspond to the potential material and bodily damage and actions are the initiatives (e.g., boarding up the house, moving to a shelter) the decision maker can take to mitigate the damage. Bets are alternative insurance policies and observations are weather forecasts. The uncertainty in this example is resolved in two stages. In the first stage a weather forecast obtains, upon the receipt of which, an action and a bet, prescribed by the strategy, are put into effect. In the second stage, the path and force of the hurricane are determined, the associated damage is realized and insurance payoff is affected.

Consider next the issue of subjective probabilities. There are two aspects of uncertainty about which, presumably, the decision maker entertains beliefs at the point at which he contemplates his strategies. The first concerns the likelihoods of alternative weather reports and, conditional on these reports, the likelihoods of subsequent paths-force combinations of the approaching hurricane. The second is the likelihoods of the ensuing levels of damage (the effects). Clearly, likelihoods of the latter is determined by those of the former coupled with the actions that were taken, in the interim, by the decision maker.

Karni (2009) deals solely with the second aspect of uncertainty and beliefs. One objective of this paper is to extend the analytical framework of Karni (2009) to include a state space and define a unique, choice-based, subjective probability measure on the state space generating the family of joint probability distributions $\{\pi(\cdot,\cdot \mid a)\}_{a\in A}$ that figure in the representation. The uniqueness of the subjective probability measure on the state space resolves the fundamental difficulty with the definition of subjective probabilities mentioned above.

A second objective is to generalize the model, allowing for action-bet interaction and, consequently, the possibility that the decision maker's risk attitudes may be affected by his choice of action. In other words, the representation (1) is replaced by the more general form

$$I \mapsto \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi\left(\theta, x \mid a_{I(x)}\right) \widetilde{u}\left(a_{I(x)}, b_{I(x)}\left(\theta\right), \theta\right)), \tag{2}$$

where the utility functions $\{\widetilde{u}\left(a, b\left(\theta\right), \theta\right)\}_{\theta \in \Theta}$ are not necessarily separately additive over actions and bets. Under this generalization the model can be applied to the analysis of decision involving actions that have monetary dimensions (e.g., protecting a property against theft by installing an alarm system). It requires two main changes: the axiom of independent betting preferences of Karni (2009) is weakened to include action-dependent betting preferences and a new concept – strings of constant-utility bets – is introduced and incorporated into the analysis.

To attain these objectives, certain modifications of the original model are unavoidable. However, the approach remains choice-based and Bayesian. The choice-based aspect maintains that a decision-maker's choice among alternative courses of action reflects his tastes for the ultimate outcomes and his beliefs regarding the likelihoods of the events in which these payoffs materialize. Consequently, the utility representing the decision maker's tastes and the probabilities representing his beliefs may be inferred from his choice behavior. The Bayesian aspect of the model takes as premises that: (a) new information affects the decision maker's preferences, or choice behavior, through its effect on his beliefs rather than his tastes, and (b) the posterior probabilities, representing the decision maker's posterior beliefs, are obtained by the updating the prior probabilities, representing his prior beliefs, using Bayes' rule.

The paper consists of two main parts. Section 2 describes the analytical framework, including the definition of a state-space, the preference structure, and the main representation theorem. Section 3 defines the measure space implied by the model and characterizes the unique probability measure on this space that generate the family of action-dependent subjective probability distributions on the effects introduced in Section 2. Section 4 includes concluding remarks. The proofs are collected in the appendix.

2 The Model

2.1 The analytical framework

Following Karni (2009), let Θ be a finite set of *effects;* let A be a connected separable topological space, whose elements are referred to as *actions*; let X a finite set of *observations;* denote by o the event that no observation materializes and define $\bar{X} = X \cup \{o\}$.[4] A *bet* is a real-valued mapping on Θ, interpreted as monetary payoffs contingent on the realized effect. Let B denote the set of all bets and assume that it is endowed with the $\mathbb{R}^{|\Theta|}$ topology. Denote by $b_{-\theta}r$ the bet obtained from $b \in B$ by replacing the θ-coordinate of b, $b(\theta)$, with r.

Informative and noninformative signals in the form of observation may be received by the decision maker before he chooses a bet and an action, and affect his choice. The decision maker is supposed to formulate a strategy specifying the action-bet pairs to be implemented

[4]The interpreation of these terms is as in the introduction.

contingent on the observations. Formally, a *strategy* is a function $I : \bar{X} \rightarrow A \times B$ whose interpretation is a set of instructions specifying, for each informational event an action-bet pair, $I(x)$, to be implemented if the informational event x obtains. Let $\mathcal{I}$ denote the set of all strategies.

A decision maker is characterized by a preference relation $\succcurlyeq$ on $\mathcal{I}$. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succcurlyeq$, respectively.

As usual, a consequence depicts those aspects of the decision problem that affect the decision maker's ex-post well-being. In this model, a *consequence* is a triplet (a, r, θ) representing, respectively, the action, the monetary payoff of the bet, and the effect. The set of all consequences is given by the Cartesian product $C = A \times \mathbb{R} \times \Theta$.

A *state of nature*, or a *state*, is a complete resolution of uncertainty, "a description of the world so complete that, if true and known, the consequences of every action would be known" (Arrow [1981], p. 45). Thus a state s is a function from $\mathcal{I}$ to C. The set $S := \{s : \mathcal{I} \rightarrow C\}$ is the *state space.* Subsets of S are *events.* One of the elements of S is the *true* state. An event is said to *obtain* if the true state is an element of it.

Uncertainty in this model is resolved in two stages. In the interim stage, an observation, $x \in \bar{X}$, obtains and the action and bets prescribed by the strategies for that observation are implemented. In the second stage, the effect is realized and the payoffs of the bet are made. Let Ω be the set of all functions from the set of actions to the set of effects (that is, $\Omega := \{\omega : A \to \Theta\}$). Elements of Ω depict the resolution of uncertainty surrounding the effects. Thus $S = \bar{X} \times \Omega$, and each state $s = (x, \omega)$ is an intersection of an *informational event* $\{x\} \times \Omega$ and a *material event* $\bar{X} \times \{\omega\}$. In other words, a state has two distinct dimensions corresponding to the two stages of the resolution of uncertainty, the purely informational dimension, x, and the possibly substantive dimension, ω. The informational event does not affect the decision maker's well-being directly whereas the material event may.

In general, states are abstract representations of the resolution of uncertainty. In some situations, however, it is natural to attribute a concrete interpretation to the states. In the example of the hurricane in the introduction, the informational events are weather forecasts, and the material events correspond to specific physical phenomena, namely, the path and force of the hurricane. Another example, due to Luce and Krantz (1971), envisions a passenger who, to get from here to there, must choose among driving, taking a bus, or flying . Whether and when he arrives at his destination depends on conditions beyond his control, such as the weather, the mechanical functioning of the alternative means of transportation, road congestion, and so forth. Conceivably, before choosing the means of transportation and placing a bet on the outcome (for example, by taking out insurance), the passenger may get some relevant information (e.g., weather forecast, FAA report on near misses at the destination airport, road construction along his route, etc.) that may affect his decision. The uncertainty regarding the outcome of the trip is resolved once the weather forecast is obtained and the mechanical functioning of the alternative means of transportation, roads congestion, whether there has been a plane crash, and so forth become known.

A main concern of this paper is to define a $\sigma-$algebra, $\mathcal{E}$, on S and a unique probability measure, P, on the measurable space $(S, \mathcal{E})$ such that (a) the conditioning of P on the noninformative signal o represents the decision maker's prior beliefs and (b) the conditioning of P on informative signals $x \in X$ represent the decision maker's posterior beliefs.

Denote by $I_{-x}(a,b)$ the strategy in which the x-coordinate of I, $I(x)$, is replaced by (a,b). The truncated strategy I_{-x} is referred to as a substrategy. For every given $x \in \bar{X}$, denote by $\succcurlyeq^x$ the induced preference relation on $A \times B$ defined by $(a,b) \succcurlyeq^x (a',b')$ if and only if $I_{-x}(a,b) \succcurlyeq I_{-x}(a',b')$. The induced strict preference relation, denoted by $\succ^x$, and the induced indifference relation, denoted by $\sim^x$, are the asymmetric and symmetric parts of $\succcurlyeq^x$, respectively.[5] The induced preference relation $\succcurlyeq^o$ is referred to as the *prior* preference relation; the preference relations $\succcurlyeq^x, x \in X$, are the *posterior* preference relations. An observation, x, is *essential* if $(a,b) \succ^x (a',b')$ for some $(a,b),(a',b') \in A \times B$. I assume throughout that all elements of $\bar{X}$ are essential.

For every $a \in A$ and $x \in \bar{X}$, define a binary relation $\succcurlyeq_a^x$ on B by: for all $b, b' \in B$, $b \succcurlyeq_a^x b'$ if and only if $(a,b) \succcurlyeq^x (a,b')$. The asymmetric and symmetric parts of $\succcurlyeq_a^x$ are denoted by $\succ_a^x$ and $\sim_a^x$, respectively.

An effect, θ, is said to be *nonnull given the observation-action pair* (x,a) if $(b_{-\theta}r) \succ_a^x (b_{-\theta}r')$, for some $b \in B$ and $r, r' \in \mathbb{R}$; it is *null given the observation-action pair* (x,a)

[5] For preference relations satisfaying (A.1) - (A.3) below, these relations are well-defined. In particular, they are independent of I.

otherwise. Given a preference relation, $\succcurlyeq$, denote by $\Theta(a,x)$ the subset of effects that are nonnull given the observation-action pair (x,a). Assume that $\Theta(a,o) = \Theta$, for all $a \in A$. Note that if $\omega(a) \notin \Theta(a,x)$, then the state (x,ω) is null.

2.2 The preference structure

With slight variations in axioms (A.4), (A.6), and (A.7), all the axioms below were introduced, and their meaning discussed, in Karni (2009). I therefore refrain from further elaboration here.

Consider the following axioms depicting the structure of a preference relation $\succcurlyeq$ on $\mathcal{I}$.

(A.1) (**Weak order**) $\succcurlyeq$ *is a complete and transitive binary relation.*

A topology on $\mathcal{I}$ is needed to define continuity of the preference relation $\succcurlyeq$. Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$, and let $\mathcal{I}$ be endowed with the product topology.[6]

(A.2) (**Continuity**) *For all* $I \in \mathcal{I}$, *the sets* $\{I' \in \mathcal{I} \mid I' \succcurlyeq I\}$ *and* $\{I' \in \mathcal{I} \mid I \succcurlyeq I'\}$ *are closed.*

The next axiom, coordinate independence, is analogous to but weaker than Savage's

[6]Recall that A is a topological space and assume that B is endowed with the $\mathbb{R}^n$ topology. Then the topology on $\mathcal{I}$ is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

(1954) sure-thing principle.[7] Like the sure-thing principle, it requires that strategies be compared by the aspects (coordinates) on which they fail to agree.

(A.3) (**Coordinate independence**) *For all* $x \in \bar{X}$, $I, I' \in \mathcal{I}$, *and* $(a,b), (a',b') \in A \times B$,

$I_{-x}(a,b) \succcurlyeq I'_{-x}(a,b)$ *if and only if* $I_{-x}(a',b') \succcurlyeq I'_{-x}(a',b')$.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the observation. It is a weakening of axiom (A.4) in Karni (2009), which required, in addition, that the effect-contingent "intensity of preferences" for monetary payoffs be independent of the actions. To grasp the meaning of this axiom, note that if the payoffs were roulette lotteries a la Anscombe and Aumann (1963), then the condition would amount to the requirement that, given any action and effect, the ranking of (roulette) lotteries contingent on that action and effect be observation-independent. This would allow the decision-maker's risk attitudes to be action and effect dependent but observation independent. To avoid invoking the notion of probabilities as an primitive, it is necessary to measure the intensity of preferences in some other way.[8] To accomplish this, I extend the trade-off method of Wakker (1987). In particular, fix an action, a, an effect, θ, and an observation, x, and suppose that $(b_{-\theta}r) \sim_a^x (b'_{-\theta}r')$ and $(b'_{-\theta}r'') \sim_a^x (b_{-\theta}r''')$. This can be interpreted that, given the action, effect and observation, the "intensity of preferences" between r and r' is the same as that between r'' and r''', and they are both *measured* by

[7] See Wakker (1989) for details.

[8] In this sense, following Savage (1954), I pursue the purely subjective approach avoiding the use of probabilites as a primitives. The cardinality of the utility functions needs to be imposed by other means.

the difference between the sub-bets $b_{-\theta}$ and $b'_{-\theta}$. Now, holding the action and bet the same, consider the issue of "intensity of preferences" under another observation x' (instead of x). The axiom requires that the "intensity of preferences" between r and r' remains the same as that between r'' and r'''. In other words, if the intensity of preferences between r and r' is measured by the sub-bets $b''_{-\theta}$ and $b'''_{-\theta}$, (that is, let $\left(b''_{-\theta}r'\right) \sim_a^{x'} \left(b'''_{-\theta}r\right)$), then that between r'' and r''' must be the same, namely, $\left(b''_{-\theta}r''\right) \sim_a^{x'} \left(b'''_{-\theta}r'''\right)$. Formally,

(A.4) (**Observation-independent action-betting preferences**) *For all* $x, x' \in \bar{X}, b, b', b'', b''' \in B$, $\theta \in \Theta(x) \cap \Theta(x')$, *and* $r, r', r'', r''' \in \mathbb{R}$, *if* $(a, b_{-\theta}r) \succcurlyeq^x \left(a, b'_{-\theta}r'\right)$, $\left(a, b'_{-\theta}r''\right) \succcurlyeq^x \left(a, b_{-\theta}r'''\right)$, *and* $\left(a, b''_{-\theta}r'\right) \succcurlyeq^{x'} \left(a, b'''_{-\theta}r\right)$ *then* $\left(a, b''_{-\theta}r''\right) \succcurlyeq^{x'} \left(a, b'''_{-\theta}r'''\right)$.

To link the decision maker's prior and posterior probabilities, the next axiom asserts that, in and of itself, information is worthless. To state this axiom, let $I^{-o}(a, b)$ denote the strategy that assigns the action-bet pair (a, b) to every observation other than o (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$). The implication of adopting this strategy is that the action-bet pair to be implemented is the same, regardless the information that may be acquired. In other words, given this strategy, information is useless. The axiom requires that, given an action, the preferences on bets when new information may not be used to select the bet be the same as the preference relation conditional on no new information.

(A.5) (**Belief consistency**) *For every* $a \in A$, $I \in \mathcal{I}$ *and* $b, b' \in B$, $I_{-o}(a, b) \sim I_{-o}(a, b')$ *if and only if* $I^{-o}(a, b) \sim I^{-o}(a, b')$.

2.3 Strings of constant-utility bets

Bets whose payoffs offset the direct impact of the effects are *constant-utility bets.* Because of the weakening of (A.4), unlike in Karni (2009), in this paper the constant utility bets are not independent of the actions. This requires a modification of the analysis and a new concept, dubbed *strings of constant utility bets.*

To grasp intuition underlying the formal definition of strings of constant utility bets, it is convenient to consider first the special case in which the valuation of the bets is independent of the actions. Suppose that the bet $\widetilde{b}$ satisfies the following conditions $I_{-x}\left(a,\widetilde{b}\right) \sim I'_{-x}\left(a',\widetilde{b}\right)$ and $I_{-x}\left(a'',\widetilde{b}\right) \sim I'_{-x}\left(a''',\widetilde{b}\right)$ for some observation x, strategies I, I', and actions a, a', a'', a'''. Then, given $\widetilde{b}$ and x, the indifference $I_{-x}\left(a,\widetilde{b}\right) \sim I'_{-x}\left(a',\widetilde{b}\right)$ depicts compensating variations between the sub-strategy I_{-x} and that action a, and the sub-strategy I'_{-x} and the action a'. Similarly, the indifference $I_{-x}\left(a'',\widetilde{b}\right) \sim I'_{-x}\left(a''',\widetilde{b}\right)$ depicts compensating variations between the sub-strategy I_{-x} and that action a'', and the sub-strategy I'_{-x} and the action a'''. Hence, the difference between sub-strategies I_{-x} and I'_{-x} "measures" the difference in the intensity of preference between a and a' and also that between a'' and a'''.[9]

Recall that the choice of action affects the decision maker's well-being directly, (the disutility of action) and indirectly, through its effect on the probabilities of the alternative effects. For the second effect to be manifested, the utility must display some variation

[9]In this case, the intensity of preferences between a and a' is, in fact, the same as that between a'' and a'''.

across effects. Constant utility bets, and only constant utility bets, are distinguished by the lack of such variations. Hence, the second effect is neutralized if and only if the bet under consideration is constant utility. For such bets, solely the direct impact of the action is manifested. Because the impact of the observations on the decision maker's well-being is through the probabilities, the definition of constant utility bets requires that the intensity of preferences between any two actions be independent of the observations.

With this in mind, consider the observation x' and suppose that $I''_{-x'}\left(a, \widetilde{b}\right) \sim I'''_{-x'}\left(a', \widetilde{b}\right)$. These compensating variations imply that, given x', the measure of the intensity of preference between a and a' is the difference in the sub-strategies $I''_{-x'}$ and $I'''_{-x'}$. The definition of constant utility bets requires that the measure of the intensity of the preference between a'' and a''', *being observation-independent*, is also given by the difference in the sub-strategies $I''_{-x'}$ and $I'''_{-x'}$.

The same intuition applies to the more general case in which the impacts of the actions and bets on the decision-maker's well-being are not separable. In this instance, however, what constitute constant utility bets depend on the actions. Consequently, the difference between the sub-strategies I_{-x} and I'_{-x} measures intensity of preferences between the actions taking into account that the associated constant utility of the bets varies with the actions. Nevertheless, the crucial point remains the same, namely, when the intensity of preference between the actions and the corresponding bets is *independent of the observations,* the indirect impact of the actions and that of the observations must has been neutralized, indicating that the corresponding bets are constant utility. Formally,

Definition 1 *A mapping* $\bar{b} : A \to B$ *is* ***a string of constant-utility bets according to*** $\succcurlyeq$ *if, for all* $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ *and* $x, x' \in \bar{X}$, $I_{-x}\left(a, \bar{b}(a)\right) \sim I'_{-x}\left(a', \bar{b}(a')\right)$, $I_{-x}\left(a'', \bar{b}(a'')\right) \sim I'_{-x}\left(a''', \bar{b}(a''')\right)$ *and* $I''_{-x'}\left(a, \bar{b}(a)\right) \sim I'''_{-x'}\left(a', \bar{b}(a')\right)$ *imply* $I''_{-x'}\left(a'', \bar{b}(a'')\right) \sim I'''_{-x'}\left(a''', \bar{b}(a''')\right)$ *and* $\cap_{x \in X}\{b \in B \mid b \sim^x_a \bar{b}(a)\} = \{\bar{b}(a)\}$, *for all* $a \in A$.

To render the definition meaningful it is assumed that, given a string of constant-utility bets $\bar{b}$, for all $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$ there are $I, I', I'', I''' \in \mathcal{I}$ such that the indifferences $I_{-x}\left(a, \bar{b}(a)\right) \sim I'_{-x}\left(a', \bar{b}(a')\right)$, $I_{-x}\left(a'', \bar{b}(a'')\right) \sim I'_{-x}\left(a''', \bar{b}(a''')\right)$ and $I''_{-x'}\left(a, \bar{b}(a)\right) \sim I'''_{-x'}\left(a', \bar{b}(a')\right)$ hold.

Let $\mathcal{B}(\succcurlyeq)$ denote the set of all strings of constant-utility bets according to $\succcurlyeq$.

Definition 2 *The set* $\mathcal{B}(\succcurlyeq)$ *is* ***inclusive*** *if there is* $\bar{b} \in \mathcal{B}(\succcurlyeq)$ *such that* $(a, b) \sim^x \left(a, \bar{b}(a)\right)$, *for every* $x \in X$ *and* $(a, b) \in A \times B$.

If, for some actions, there exists no monetary compensation for the impact of the effects (that is, the ranges of the utility of the monetary payoffs across effects do not overlap), then, for that action, there is no constant utility bet and $\mathcal{B}(\succcurlyeq)$ is empty. Here I am concerned with the case in which $\mathcal{B}(\succcurlyeq)$ is inclusive, and thus nonempty.

In the special case $I = I'$ and $I'' = I''$, definition 1 implies that $\left(a, \bar{b}(a)\right) \sim^x \left(a', \bar{b}(a')\right)$ for all $x \in \bar{X}$. Anticipating the main result, this means that $\left(a, \bar{b}(a)\right)$ and $\left(a', \bar{b}(a')\right)$ correspond to the same expected utility, regardless of the observation.[10] This special case pertains,

[10] I thank Jacques Drèze for calling my attention to this special case.

naturally, to actions identified with monetary expenses that are perfect substitutes for the payoffs of the bets. In general, however, it is possible that there are no feasible monetary compensation for the disutility associated with some actions. In such a case, the expected utilities associated with $\left(a, \bar{b}\left(a\right)\right)$ and $\left(a', \bar{b}\left(a'\right)\right)$ are distinct, and the difference between them is "measured" by the utility difference between the substrategies I_{-x} and I'_{-x} as well as that between I''_{-x} and I'''_{-x}.

The next two axioms are rather straightforward. The first requires that the trade-offs between the actions and the substrategies that figure in definition 1 be independent of the constant-utility bets. Formally,

(A.6) (**Trade-off independence)** For all $I, I' \in \mathcal{I}$, $x \in \bar{X}$, $a, a' \in A$ and $\bar{b}, \bar{b}' \in \mathcal{B}\left(\succcurlyeq\right)$, $I_{-x}\left(a, \bar{b}\left(a\right)\right) \succcurlyeq I'_{-x}\left(a', \bar{b}\left(a'\right)\right)$ if and only if $I_{-x}\left(a, \bar{b}'\left(a\right)\right) \succcurlyeq I'_{-x}\left(a', \bar{b}'\left(a'\right)\right)$.

Finally, it is also required that the direct effect (that is, the cost) of actions, measured by the preferential difference between any two strings of constant-utility bets, $\bar{b}, \bar{b}' \in \mathcal{B}\left(\succcurlyeq\right)$, be independent of observation. Formally,

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b}' \in \mathcal{B}\left(\succcurlyeq\right), x, x' \in \bar{X}$, and $a, a' \in A$, $\left(a, \bar{b}\left(a\right)\right) \succcurlyeq^{x} \left(a', \bar{b}'\left(a'\right)\right)$ if and only if $\left(a, \bar{b}\left(a\right)\right) \succcurlyeq^{x'} \left(a', \bar{b}'\left(a'\right)\right)$.

2.4 Representation

The next theorem generalizes Theorem 2 of Karni (2009) by permitting interaction between actions and bets. Consequently, the effect-dependent utility functions are not necessarily separately additive in actions and bets. To simplify the statement of the results that follow, I let $I(x) = \left(a_{I(x)}, b_{I(x)}\right)$.

Theorem 3 *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $\mathcal{B}(\succcurlyeq)$ is inclusive, then*

(*a*) *The following conditions are equivalent:*

(*i*) *$\succcurlyeq$ satisfies (A.1)–(A.7).*

(*ii*) *there exist a continuous, real-valued function $\widetilde{u}$ on $A \times \mathbb{R} \times \Theta$, and a family of joint probability measures $\{\pi(\cdot,\cdot \mid a)\}_{a\in A}$ on $\bar{X} \times \Theta$ such that $\succcurlyeq$ on I is represented by*

$$I \mapsto \sum_{x\in\bar{X}} \mu(x) \sum_{\theta\in\Theta} \widetilde{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right)\pi\left(\theta \mid x, a_{I(x)}\right) \tag{3}$$

where $\mu(x) = \sum_{\theta\in\Theta} \pi(x,\theta \mid a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A$, $\pi(\theta \mid x, a) = \pi(x,\theta \mid a)/\mu(x)$ for all $x \in X$, and $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x\in X} \pi(x,\theta \mid a)$.

(*b*) *The function $\widetilde{u}$ is unique up to a positive affine transformation and, for each $a \in A$, $\pi(\cdot,\cdot \mid a)$ is unique.*

(*c*) *For every $\bar{b} \in B(\succcurlyeq)$ and $a \in A$, $\widetilde{u}\left(a, \bar{b}(a)(\theta), \theta\right) = \widetilde{u}\left(a, \bar{b}(a)(\theta'), \theta'\right)$ for all $\theta, \theta' \in \Theta$.*

Notice that, although the joint probability distributions $\pi(\cdot,\cdot \mid a), a \in A$ depend on the actions, the distribution μ is independent of a. This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information arrives, it precedes the choice of action. Hence the dependence of the joint probability distributions $\pi(\cdot,\cdot \mid a)$ on a captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.[11]

A special case of Theorem 3 obtains when actions are monetary expenditures (that is, when $A = \mathbb{R}_-$). For instance, when considering installing sprinklers to reduce the loss in case of a fire, it is natural to assume that the utility impact of this action depends solely on the money spent. Hence $\widetilde{u}(a, b(\theta), \theta)) = \widetilde{u}(a + b(\theta), \theta))$, $\theta \in \Theta$. In general, actions affect the preference directly, through their associated disutility and the possible associated "wealth effect" on the decision maker's attitudes toward the risk represented by the bets, and indirectly, through their impact on the probabilities of the effects. To isolated the "utility impact," it is necessary to confine attention to strings of constant-utility bets. The idea that, insofar as the utility is concerned, actions and bets are perfect substitutes is captured by the following axiom:

[11] If an action-bet pair are already "in effect" when new information arrives, they constitute a default course of action. In such instance, the interpretation of the decision at hand is possible choice of new action and bet. For example, a modification, upon learning the results of medical checkup, of a diet regiment coupled with a possible change of life insurance policy.

(A.8) (**Substitution**) For all $\bar{b} \in \mathcal{B}(\succcurlyeq)$, $I \in \mathcal{I}$, $x \in \bar{X}$ and $a \in \mathbb{R}_{-}, z \in \mathbb{R}$, $I_x\left(a, \bar{b}(a)\right) \sim I_x\left(a - z, \bar{b}(a) + z\right)$.

By Theorem 3 and axiom (A.8), for every $\bar{b} \in \mathcal{B}(\succcurlyeq)$ and $a \in A$, $\widetilde{u}\left(a, \bar{b}(a)(\theta), \theta\right) = \widetilde{u}\left(a - z, \bar{b}(a)(\theta) + z, \theta\right)$ for all $\theta \in \Theta$. Hence, with slight abuse of notations, $\widetilde{u}\left(a, \bar{b}(a)(\theta), \theta\right) = \widetilde{u}\left(a + \bar{b}(a)(\theta), \theta\right)$ for $\theta \in \Theta$. This implies the following:

Corollary 4 *Let $A = \mathbb{R}_{-}$ and $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $\mathcal{B}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ satisfies (A.1)–(A.8) if and only if there exist a continuous, real-valued function $\widetilde{u}$ on $A \times \mathbb{R} \times \Theta$, and a family of joint probability measures $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ on $\bar{X} \times \Theta$ such that $\succcurlyeq$ on $\mathcal{I}$ is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \sum_{\theta \in \Theta} \widetilde{u}\left(a_{I(x)} + b_{I(x)}(\theta), \theta\right))\pi\left(\theta \mid x, a_{I(x)}\right) \tag{4}$$

where $\mu(x) = \sum_{\theta \in \Theta} \pi(x, \theta \mid a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A$, $\pi(\theta \mid x, a) = \pi(x, \theta \mid a) / \mu(x)$ for all $x \in X$, and $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x, \theta \mid a)$. Moreover, $\widetilde{u}$ is unique up to positive affine transformation, for each $a \in A$ $\pi(\cdot, \cdot \mid a)$ is unique and, for every $\bar{b} \in \mathcal{B}(\succcurlyeq)$ and $a \in A$, $\widetilde{u}\left(a + \bar{b}(a)(\theta), \theta\right) = \widetilde{u}\left(a + \bar{b}(a)(\theta'), \theta'\right)$ for all $\theta, \theta' \in \Theta$.

3 Subjective Probabilities on the State Space

3.1 Action-dependent subjective probabilities on S

Let $\mathcal{Y}$ be the informational partition of S (that is, mutually disjoint informational events whose union is S). Formally, $\mathcal{Y} = \{\{x\} \times \Omega \mid x \in \bar{X}\}$.

For every $a \in A$ and $\theta \in \Theta$, let $\mathcal{T}_a(\theta) := \{\omega \in \Omega \mid \omega(a) = \theta\}$. Then $\mathcal{T}_a = \{\mathcal{T}_a(\theta) \mid \theta \in \Theta\}$ is a (finite) partition of Ω. Denote by $\mathcal{E}_a$ the $\sigma-$algebra on S generated by $\mathcal{Y} \wedge \mathcal{T}_a$, the join of $\mathcal{Y}$ and $\mathcal{T}_a$.[12] Elements of $\mathcal{E}_a$ are *events.* Hence, events are unions of elements of $\mathcal{Y} \wedge \mathcal{T}_a$. Consider the measurable spaces $(S, \mathcal{E}_a)$, $a \in A$. By Theorem 3, a preference relation $\succcurlyeq$ satisfying (A.1)–(A.7) implies the existence of a unique joint probability measure $\pi(\cdot, \cdot \mid a)$ on $\bar{X} \times \Theta$. For each $a \in A$, let η_a be a probability measure on $\mathcal{E}_a$ defined by $\eta_a(E) = \sum_{x \in Z} \sum_{\theta \in \Upsilon(x)} \pi(x, \theta \mid a)$ for every $E = \cup_{x \in Z} (\{x\} \times \mathcal{T}_a(\Upsilon(x)))$, $Z \subseteq \bar{X}$, $\Upsilon(x) \subseteq \Theta$ and $\mathcal{T}_a(\Upsilon(x)) = \cup_{\theta \in \Upsilon(x)} \mathcal{T}_a(\theta)$. The uniqueness of the conditional probability measures follows from Theorem 3. Specifically, by Theorem 3, $\eta(\{x\} \times \Omega) = \sum_{\theta \in \Theta} \pi(x, \theta \mid a)$ is independent of a. Hence, by definition, the prior probability measure on $\mathcal{E}_a$ is given by $\eta_a(\mathcal{T}_a(\theta) \mid o) = \eta_a(\{o\} \times \mathcal{T}_a(\theta)) / \eta(\{o\} \times \Omega) = \pi(o, \theta \mid a)$. Similarly, the posterior probability measures on $\mathcal{E}_a$ are given by $\eta_a(\mathcal{T}_a(\theta) \mid x) = \eta_a(\{x\} \times \mathcal{T}_a(\theta)) / \eta(\{x\} \times \Omega) = \pi(x, \theta \mid a)$, for every $x \in X$. The uniqueness of the prior and posteriors follow from the uniqueness of π. Moreover,

Theorem 3 may be restated in terms of these probability measures as follows:

[12] The join of two partitions is the coarsest common refinement of these partitions.

Corollary 5 *Let $\succcurlyeq$ be a preference relation on $\mathcal{I}$ and suppose that $\mathcal{B}(\succcurlyeq)$ is inclusive. Then $\succcurlyeq$ satisfies (A.1)-(A.7) if and only if there exist a continuous, real-valued function $\widetilde{u}$ on $A \times \mathbb{R} \times \Theta$ and, for each $a \in A$, a unique probability measures η_a on the measurable space $(S, \mathcal{E}_a)$ such that $\eta_a(\{x\} \times \Omega) = \eta_{a'}(\{x\} \times \Omega) = \eta(x)$ for all $a, a' \in A$ and $\succcurlyeq$ on $\mathcal{I}$ is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \eta(x) \sum_{\theta \in \Theta} \widetilde{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) \eta_{a_{I(x)}}\left(\mathcal{T}_{a_{I(x)}}(\theta) \mid x\right). \tag{5}$$

Moreover, $\widetilde{u}$ is unique up to positive affine transformation.

Corollary 5 asserts the existence of a unique family of action-dependent prior and posterior probability measures on the measurable spaces $\{(S, \mathcal{E}_a)\}_{a \in A}$.[13] This collection of measure spaces is sufficiently rich to allow an action-dependent probability to be defined for every event that matters to the decision maker, given all the conceivable choices among strategies that he might be called upon to make. Hence from the point of view of Bayesian decision theory, the family of action-dependent subjective probability measures defined here is complete, in the sense of being well defined for every conceivable decision problem that can be formulated in this framework. However, there is no guarantee that these subjective probability measures are mutually consistent. Hence it interesting to inquire about the necessary

[13] Notice that the utility functions in the representation could have been written in terms of the *material events* as follows:

$$I \mapsto \sum_{x \in \bar{X}} \eta(x) \sum_{\theta \in \Theta} \widetilde{u}\left(a_{I(x)}, b_{I(x)}\left(\mathcal{T}_{a_{I(x)}}(\theta)\right), \mathcal{T}_{a_{I(x)}}(\theta)\right) \eta_{a_{I(x)}}\left(\mathcal{T}_{a_{I(x)}}(\theta) \mid x\right)$$

and sufficient conditions for the existence of a unique probability space that supports all these action-dependent measures in the sense that $\eta_a(E)$ coincides with this measure on the events $E \in \mathcal{E}_a$. This issue is taken up next.

3.2 Subjective probabilities on S

To begin with, it is necessary to define a σ-algebra that includes the class of events $\cup_{a \in A}\mathcal{E}_a$.[14] Let $\wedge_{i=1}^{k}\mathcal{T}_{a_i}$ be the join of $\mathcal{T}_{a_1}, ..., \mathcal{T}_{a_k}$, and denote by $\mathcal{E}$ the σ-algebra generated by $\{\wedge_{i=1}^{k}\mathcal{T}_{a_i} \mid k < \infty\} \wedge \mathcal{Y}$. The issue posed at the end of the last subsection may be restated formally as follows: What are the necessary and sufficient conditions for the existence of a unique probability measure P on the measurable space $(S, \mathcal{E})$, such that $P(E) = \eta_a(E)$ for all $E \in \cup_{a \in A}\mathcal{E}_a$, and $P(\{x\} \times \Omega) = \eta(\{x\} \times \Omega)$?

This question raises two issues. The first, mentioned above, concerns the consistency of beliefs across actions. To illustrate this issue and motivate the ensuing inquiry, I consider the example of the approaching hurricane described in the introduction. In this example, the substantive aspect of the state is concrete, namely, the force and path of the approaching hurricane. The effects of the storm are determined by the action taken, a, and the force and path of the hurricane, ω. In this case, the conditional probability of the effects conditional on the action, and the observation, $\pi(\theta, x \mid a)$ corresponds to probability $\eta_a(E)$, where $E = \{x\} \times \mathcal{T}_a(\theta)$. It is also intuitively clear that, if $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ then the probability

[14]Note that $\cup_{a \in A}\mathcal{E}_a$ itself is not a $\sigma-$ algebra.

that E obtains, (that is, that the true state consists of the forecast x and that the force and trajectory of the hurricane is a point in the set $\mathcal{T}_a(\theta)$ are independent of the action. Formally, if $\{x\} \times \mathcal{T}_{a'}(\theta') = \{x\} \times \mathcal{T}_a(\theta) = E$, then, because the probabilities of the effects are induced by the probabilities of the underlying state, consistency requires that $\pi(\theta, x \mid a) = \eta_a(E) = \eta_{a'}(E) = \pi(\theta', x \mid a')$.

The second issue arises because a probability measure on the measurable space $(S, \mathcal{E})$ requires that the probabilities of events such as $\mathcal{T}_a(\theta) \cap \mathcal{T}_{a'}(\theta') \neq \varnothing$ for some $\theta, \theta' \in \Theta$ and distinct actions $a, a' \in A, .bewell - deined$ In general, the probabilities of such events may not be determined from the probabilities of the events in the collection $\cup_{a \in A} \mathcal{E}_a$.

These issues are addressed in next two subsections.

3.2.1 Belief consistency

For each $a \in A$, $\bar{b}(a), \bar{b}'(a) \in \mathcal{B}$, and $E \in \mathcal{E}_a$, let $\bar{b}(a)_E \bar{b}'(a) \in B$ be defined by

$$\bar{b}(a)_E \bar{b}'(a)(\theta) = \left\{ \begin{array}{lll} \bar{b}(a)(\theta) & \text{if} & \mathcal{T}_a(\theta) \subseteq E \\ \bar{b}'(a)(\theta) & \text{if} & \mathcal{T}_a(\theta) \subseteq \Omega - E \end{array} \right\}.$$

Thus, $\bar{b}(a)_E \bar{b}'(a)$ is the bet whose utility payoffs is a doubleton, paying $\widetilde{u}\left(a, \bar{b}(a)\right)$ for all θ such that $\mathcal{T}_a(\theta) \subset E$ and $\widetilde{u}\left(a, \bar{b}'(a)\right)$ for all θ such that $\mathcal{T}_a(\theta) \subset \Omega - E$. If $\bar{b}(a) > \bar{b}'(a)$, then $\bar{b}(a)_E \bar{b}'(a)$ is a bet on E.

Definition 6 *: For all $E \in \cup_{a \in A} \mathcal{E}_a$, $a, a' \in A$, and $\bar{b}, \bar{b}' \in \mathcal{B}(\succcurlyeq)$, if $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ then E is* ***believed equally likely to obtain under** a **and under** a'* *given $x \in \bar{X}$, if $I_{-x}\left(a, \bar{b}(a)_E \bar{b}'(a)\right) \sim$*

$I'_{-x}\left(a', \bar{b}\left(a'\right)_E \bar{b}'\left(a'\right)\right)$, *where* $I_{-x}\left(a, \bar{b}\left(a\right)\right) \sim I'_{-x}\left(a', \bar{b}\left(a'\right)\right)$, $I_{-x}\left(a, \bar{b}'\left(a\right)\right) \sim I'_{-x}\left(a', \bar{b}'\left(a'\right)\right)$ *and* $I_{-x}\left(a, \bar{b}\left(a\right)\right) \succ I_{-x}\left(a, \bar{b}'\left(a\right)\right)$.

Let this (partial) binary relation on $\cup_{a \in A} \mathcal{E}_a$ be denoted by $(a, E) \sim_L^x (a', E)$, then $\sim_L^x$ is an equivalence. The axiom requires that if the same event may obtain under two distinct actions, then the beliefs about its likelihood be action independent. Formally,

(A.9) (**Belief consistency**) For all $a, a' \in A$ and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$, $(a, E) \sim_L^x (a', E)$.

The following theorem asserts that, under belief consistency, the action-dependent subjective probability measures, $\{\eta_a\}_{a \in A}$, agree on events in the intersections of the σ-algebras $\{\mathcal{E}_a\}_{a \in A}$.

Theorem 7 *Let* $\succcurlyeq$ *be a preference relation on* $\mathcal{I}$ *satisfying (A.1)–(A.7), and suppose that* $\mathcal{B}(\succcurlyeq)$ *is inclusive. Then* $\succcurlyeq$ *satisfies belief consistency, (A.9), if and only if* $\eta_a(E) = \eta_{a'}(E)$, *for all* $a, a' \in A$ *and* $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$.

3.2.2 Richness

Consider next the problem posed by events in $\mathcal{E}$ that are in the intersection of elements of the partitions of the state space under distinct actions. In general, the probabilities of such events cannot be inferred from the action-dependent joint probabilities of observations and effects that figure in Theorem 3. This problem can be surmounted if the model is rich in the sense that, for every finite collection of actions and corresponding subsets of effects, the event that yields these effects under these actions coincides with an event in $\mathcal{E}_a$ for some $a \in A$. Formally,

Definition 8 *The model* $\left(S, A, \Theta, \bar{X}\right)$ *is* ***rich*** *if, for all* $(a_i, \Upsilon_i) \in A \times 2^{\Theta}$, $i = 1, ..., k$, *and* $x \in \bar{X}$, $\{x\} \times \cap_{i=1}^{k} \mathcal{T}_{a_i}(\Upsilon_i) = E$ *for some* $E \in \cup_{a \in A} \mathcal{E}_a$.

The next theorem establishes the existence and uniqueness of a probability space supporting the action-dependent distributions. Let $P(x) := P(\{x\} \times \Omega)$.

Theorem 9 *Let* $\succcurlyeq$ *be a preference relation on* $\mathcal{I}$. *Suppose that* $\mathcal{B}(\succcurlyeq)$ *is inclusive and that the model* $\left(S, A, \Theta, \bar{X}\right)$ *is rich. Then* $\succcurlyeq$ *satisfies (A.1)–(A.7) and (A.9) if and only if there exist a continuous, real-valued function* $\widetilde{u}$ *on* $A \times \mathbb{R} \times \Theta$ *and a unique probability measure* P *on* $(S, \mathcal{E})$ *such that* $\succcurlyeq$ *on* $\mathcal{I}$ *is represented by*

$$I \mapsto \sum_{x \in \bar{X}} P(x) \sum_{\theta \in \Theta} \widetilde{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) P\left(\mathcal{T}_{a_{I(x)}}(\theta) \mid x\right). \tag{6}$$

Moreover, $\widetilde{u}$ *is unique up to positive affine transformation.*

The proof invokes the probability distributions of Theorem 3 and the richness of the model to define a probability measure on the algebra generated by $\{\wedge_{i=1}^{k} \mathcal{T}_{a_i} \mid k < \infty\} \wedge \mathcal{Y}$. This probability measure has a unique extension to $(S, \mathcal{E})$.

4 Concluding Remarks

This paper presents a complete model of Bayesian decision making providing a choice-based foundations of prior and posterior subjective probabilities on a state-space. From the point of view of the theory of decision making under uncertainty, this model is quite general, allowing for effect-dependent and action-dependent utility functions of wealth. This feature makes it applicable for the analysis of life and health insurance problems, medical decision making, and other decisions in which the effect interacts with the decision maker's wealth. Moreover, by allowing the for interaction between the decision maker's actions and the evaluation of his wealth. This is a crucial aspect when the actions themselves include monetary expenses. Special instances of the theory presented here, including representations of preference relations displaying effect-independence preferences, may be obtained in a straightforward manner following Karni (2009). In addition, this paper furnishes an axiomatic foundation of the agent's behavior in the state-space formulation of the principal-agent problem.[15]

The definition of unique, choice-based, subjective probabilities on a state-space settles a long standing issue in Bayesian theory.

[15]See Hart and Holmstrom (1987)

APPENDIX

For expository convenience, I write $\mathcal{B}$ instead of $\mathcal{B}(\succcurlyeq)$.

4.1 Proof of Theorem 3

(a) (Sufficiency) Assume that $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)–(A.7) and $\mathcal{B}$ is inclusive. Let $\mathcal{I}$ be endowed with the product topology and suppose that $|\bar{X}| \geq 3$.[16]

By Wakker (1989) Theorem III.4.1, $\succcurlyeq$ satisfies (A.1)–(A.3) if and only if there exist an array of real-valued functions $\{w(\cdot,\cdot,x) \mid x \in \bar{X}\}$ on $A \times B$ such that $\succcurlyeq$ is represented by

$$I \mapsto \sum_{x \in \bar{X}} w\left(a_{I(x)}, b_{I(x)}, x\right), \tag{7}$$

where $w(.,.,x), x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.[17]

Since $\succcurlyeq$ satisfies (A.4), Lemmas 5 and 6 in Karni (2006) and Theorem III.4.1 in Wakker (1989) imply that, for every $x \in \bar{X}$ and $a \in A$ such that $\Theta(a,x)$ contains at least two effects,

[16] To simplify the exposition I state the theorem for the case in which $\bar{X}$ contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker [1989] theorem III.4.1).

[17] An array of real-valued functions $(v_s)_{s \in S}$ is said to be a *jointly cardinal additive representation* of a binary relation $\succeq$ on a product set $D = \Pi_{s \in S} D_s$ if, for all $d, d' \in D$, $d \succeq d'$ if and only if $\sum_{s \in S} v_s(d_s) \geq \sum_{s \in S} v_s(d'_s)$, and the class of all functions that constitute an additive representation of $\succeq$ consists of those arrays of functions, $(\hat{v}_s)_{s \in S}$, for which $\hat{v}_s = \lambda v_s + \zeta_s$, $\lambda > 0$ for all $s \in S$. The representation is continuous if the functions $v_s, s \in S$ are continuous.

there exist array of functions $\{v_x(a,\cdot;\theta):\mathbb{R}\to\mathbb{R}\mid\theta\in\Theta\}$ constituting a jointly cardinal, continuous, additive, representation of $\succcurlyeq_a^x$ on B. Moreover, by the proof of Lemma 6 in Karni (2006), $\succcurlyeq$ satisfies (A.1)–(A.4) if and only if, for every $x,x'\in\bar{X}$ and $a\in A$ satisfying $\Theta(a,x)\cap\Theta(a,x')\neq\varnothing$ and $\theta\in\Theta(a,x)\cap\Theta(a,x')$, there exist $\beta_{(x',x,a,\theta)}>0$ and $\alpha_{(x',x,a,\theta)}$ such that $v_{x'}(a,\cdot,\theta)=\beta_{(x',x,a,\theta)}v_x(a,\cdot,\theta)+\alpha_{(x',x,a,\theta)}$.[18]

Define $u(a,\cdot,\theta)=v_o(a,\cdot,\theta)$, $\lambda(a,x;\theta)=\beta_{(x,o,a,\theta)}$ and $\alpha(a,x,\theta)=\alpha_{(x,o,a,\theta)}$ for all $a\in A$, $x\in\bar{X}$, and $\theta\in\Theta$. For every given $x\in\bar{X}$ and $a\in A$, $w(a,b,x)$ represents $\succcurlyeq_a^x$ on B. Hence

$$w(a,b,x)=H\left(\sum_{\theta\in\Theta}\left(\lambda(a,x,\theta)\,u(a,b(\theta);\theta)+\alpha(a,x,\theta)\right),a,x\right),\tag{8}$$

where H is a continuous, increasing function.

Consider next the restriction of $\succcurlyeq$ to $L:=\{(a,\bar{b}(a))\mid a\in A,\bar{b}\in\mathcal{B}\}^{\bar{X}}$.

Lemma 10 *There exist functions $U:L\to\mathbb{R}$, $\xi:\bar{X}\to\mathbb{R}_{++}$, and $\zeta:\bar{X}\to\mathbb{R}$ such that, for all $(a,\bar{b},x)\in A\times\mathcal{B}\times\bar{X}$,*

$$w\left(a,\bar{b}(a),x\right)=\xi(x)\,U\left(a,\bar{b}(a)\right)+\zeta(x),\ \ \xi(x)>0.\tag{9}$$

Proof: Let $I,I',I'',I'''\in\mathcal{I}$, $a,a',a'',a'''\in A$ and $\bar{b}(a),\bar{b}(a'),\bar{b}(a''),\ \bar{b}(a''')$ be as in definition 1. Then, for all $x,x'\in\bar{X}$, $I_{-x}\left(a,\bar{b}(a)\right)\sim I'_{-x}\left(a',\bar{b}(a')\right)$, $I_{-x}\left(a'',\bar{b}(a'')\right)\sim I'_{-x}\left(a''',\bar{b}(a''')\right)$, $I''_{-x'}\left(a,\bar{b}(a)\right)\sim I'''_{-x'}\left(a',\bar{b}(a')\right)$ and $I''_{-x'}\left(a'',\bar{b}(a'')\right)\sim I'''_{-x'}\left(a''',\bar{b}(a''')\right)$. By

[18] By definition, for all (a,x) and θ, $\beta_{(x,x,a,\theta)}=1$ and $\alpha_{(x,x,a,\theta)}=0$.

the representation (7), $I_{-x}\left(a,\bar{b}\left(a\right)\right)\sim I'_{-x}\left(a',\bar{b}\left(a'\right)\right)$ implies that

$$\sum_{y\in\bar{X}-\{x\}} w\left(a_{I(y)},b_{I(y)},y\right)+w\left(a,\bar{b}\left(a\right),x\right)=\sum_{y\in\bar{X}-\{x\}} w\left(a_{I'(y)},b_{I'(y)},y\right)+w\left(a',\bar{b}\left(a'\right),x\right). \quad (10)$$

Similarly, $I_{-x}\left(a'',\bar{b}\left(a''\right)\right)\sim I'_{-x}\left(a''',\bar{b}\left(a'''\right)\right)$ implies that

$$\sum_{y\in\bar{X}-\{x\}} w\left(a_{I(y)},b_{I(y)},y\right)+w\left(a'',\bar{b}\left(a''\right),x\right)=\sum_{y\in\bar{X}-\{x\}} w\left(a_{I'(y)},b_{I'(y)},y\right)+w\left(a''',\bar{b}\left(a'''\right),x\right), \quad (11)$$

$I''_{-x'}\left(a,\bar{b}\left(a\right)\right)\sim I'''_{-x'}\left(a',\bar{b}\left(a'\right)\right)$ implies that

$$\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I''(y)},b_{I''(y)},y\right)+w\left(a,\bar{b}\left(a\right),x'\right)=\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I'''(y)},b_{I'''(y)},y\right)+w\left(a',\bar{b}\left(a'\right),x'\right), \quad (12)$$

and $I''_{-x'}\left(a'',\bar{b}\left(a''\right)\right)\sim I'''_{-x'}\left(a''',\bar{b}\left(a'''\right)\right)$ implies that

$$\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I''(y)},b_{I''(y)},y\right)+w\left(a'',\bar{b}\left(a''\right),x'\right)=\sum_{y\in\bar{X}-\{x'\}} w\left(a_{I'''(y)},b_{I'''(y)},y\right)+w\left(a''',\bar{b}\left(a'''\right),x'\right). \quad (13)$$

But (10) and (11) imply that

$$w\left(a,\bar{b}\left(a\right),x\right)-w\left(a',\bar{b}\left(a'\right),x\right)=w\left(a'',\bar{b}\left(a''\right),x\right)-w\left(a''',\bar{b}\left(a'''\right),x\right). \quad (14)$$

and (12) and (13) imply that

$$w\left(a,\bar{b}\left(a\right),x'\right)-w\left(a',\bar{b}\left(a'\right),x'\right)=w\left(a'',\bar{b}\left(a''\right),x'\right)-w\left(a''',\bar{b}\left(a'''\right),x'\right). \quad (15)$$

Define a function $\phi_{\left(x,x',\bar{b}\right)}$ as follows: $w\left(\cdot,\bar{b}\left(\cdot\right),x\right)=\phi_{\left(x,x',\bar{b}\right)}\circ w\left(\cdot,\bar{b}\left(\cdot\right),x'\right)$. Then $\phi_{\left(x,x',\bar{b}\right)}$ is continuous. Axiom (A.7) implies that $\phi_{\left(x,x',\bar{b}\right)}$ is monotonic increasing. Moreover, equations (14) and (15) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{\left(x,x',\bar{b}\right)}$ is affine.

Let $\beta_{(x,o,\bar{b})} > 0$ and $\delta_{(x,o,\bar{b})} := \zeta(x)$ denote, respectively, the multiplicative and additive coefficients corresponding to $\phi_{(x,o,\bar{b})}$, where the inequality follows from the monotonicity of $\phi_{(x,o,\bar{b})}$. Observe that $I_{-o}\left(a,\bar{b}(a)\right) \sim I'_{-o}\left(a',\bar{b}(a')\right)$ and $I_{-o}\left(a,\bar{b}'(a)\right) \sim I'_{-o}\left(a',\bar{b}'(a')\right)$ in conjunction with axiom (A.6) imply that

$$\beta_{(x,o,\bar{b})}\left[w\left(a,\bar{b}(a),o\right) - w\left(a',\bar{b}(a'),o\right)\right] = \beta_{(x,o,\bar{b}')}\left[w\left(\cdot,\bar{b}'(a),o\right) - w\left(a',\bar{b}'(a'),o\right)\right] \quad (16)$$

for all $\bar{b},\bar{b}' \in \mathcal{B}$. Thus, for all $x \in \bar{X}$ and $\bar{b},\bar{b}' \in \mathcal{B}$, $\beta_{(x,o,\bar{b})} = \beta_{(x,o,\bar{b}')} := \xi(x) > 0$.

Let $a, a' \in A$ and $\bar{b},\bar{b}' \in \mathcal{B}$ satisfy $\left(a,\bar{b}(a)\right) \sim^{o} \left(a',\bar{b}'(a')\right)$. By axiom (A.7) $\left(a,\bar{b}(a)\right) \sim^{x} \left(a',\bar{b}'(a')\right)$ if and only if $\left(a,\bar{b}(a)\right) \sim^{o} \left(a',\bar{b}'(a')\right)$. By the representation this equivalence implies that

$$w\left(a,\bar{b}(a),o\right) = w\left(a',\bar{b}'(a'),o\right). \quad (17)$$

if and only if,

$$\xi(x)\, w\left(a,\bar{b}(a),o\right) + \delta_{(x,o,\bar{b})} = \xi(x)\, w\left(a',\bar{b}'(a'),o\right) + \delta_{(x,o,\bar{b}')}. \quad (18)$$

Thus $\delta_{(x,o,\bar{b})} = \delta_{(x,o,\bar{b}')}$. By continuity, (A.2), the conclusion can be extended to $\mathcal{B}$. Let $\delta_{(x,o,\bar{b})} := \zeta(x)$ for all $\bar{b} \in \mathcal{B}$.

For all $a \in A$ and $\bar{b} \in \mathcal{B}$, define $U\left(a,\bar{b}(a)\right) = w\left(a,\bar{b}(a),o\right)$ and let $\xi(x)$ and $\zeta(x)$ denote the multiplicative and additive part of $\phi_{(x,o,\bar{b})}$. Then, for all $x \in \bar{X}$,

$$w\left(a,\bar{b}(a),x\right) = \xi(x)\, U\left(a,\bar{b}(a)\right) + \zeta(x),\ \ \xi(x) > 0. \quad (19)$$

This completes the proof of Lemma 10. ♣

Let $\hat{\alpha}(a,x) = \sum_{\theta\in\Theta} \alpha(a,x,\theta)$, then equations (8) and (9) imply that for every $x \in \bar{X}$, $\bar{b} \in \mathcal{B}$ and $a \in A$,

$$\xi(x) U\left(a, \bar{b}(a)\right) + \zeta(x) = H\left(\sum_{\theta\in\Theta} \lambda(a,x,\theta) u\left(a, \bar{b}(a)(\theta), \theta\right) + \hat{\alpha}(a,x), a, x\right). \tag{20}$$

Lemma 11 *The identity (20) holds if and only if* $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a, \bar{b}(a), \theta'\right)$ *for all* $\theta, \theta' \in \Theta$, $\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} = \varphi(a)$, $\frac{\hat{\alpha}(a,x)}{\xi(x)} = v(a)$ *for all* $a \in A$,

$$\xi(x)\left[\varphi(a) u\left(a, \bar{b}(a)\right) + v(a)\right] + \zeta(x) = H\left(\sum_{\theta\in\Theta} \lambda(a,x,\theta) u\left(a, \bar{b}(a)(\theta), \theta\right) + \hat{\alpha}(a,x), a, x\right), \tag{21}$$

and

$$\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)} = U\left(a, \bar{b}(a)\right). \tag{22}$$

Proof: (Sufficiency) Let $u\left(a, \bar{b}(a)(\theta), \theta\right) := u\left(a, \bar{b}(a)\right)$ for all $\theta \in \Theta$, $\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} := \varphi(a)$ and $\frac{\hat{\alpha}(a,x)}{\xi(x)} = v(a)$ for all $a \in A$ and suppose that (22) holds. Then equation (20) follows from equation (21).

(Necessity) Multiply and divide the first argument of H by $\xi(x) > 0$. Equation (20) may be written as follows:

$$\xi(x) U\left(a, \bar{b}(a)\right) + \zeta(x) = H\left(\xi(x)\left[\sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}\right], a, x\right). \tag{23}$$

Define $V\left(a, \bar{b}(a), x\right) = \sum_{\theta\in\Theta} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}$ then, for every given $(a,x) \in$

$A \times X$ and all $\bar{b}, \bar{b}' \in \mathcal{B}$,

$$U\left(a, \bar{b}(a)\right) - U\left(a, \bar{b}'(a)\right) = \left[H\left(\xi(x) V\left(a, \bar{b}(a), x\right), a, x\right) - H\left(\xi(x) V\left(a, \bar{b}'(a), x\right), a, x\right)\right] / \xi(x). \tag{24}$$

Hence $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$\left[U\left(a, \bar{b}(a)\right) - U\left(a, \bar{b}'(a)\right)\right] / \left[V\left(a, \bar{b}(a), x\right) - V\left(a, \bar{b}'(a), x\right)\right] := \kappa(a),$$

is independent of x. Thus

$$\xi(x) U\left(a, \bar{b}(a)\right) + \zeta(x) = \kappa(a) \xi(x) \left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)}\right] + \zeta(x). \tag{25}$$

Hence

$$U\left(a, \bar{b}(a)\right) / \kappa(a) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(a, \bar{b}(a)(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \tag{26}$$

is independent of x. However, because $\succcurlyeq_a^x \neq \succcurlyeq_a^{x'}$ for all a and some $x, x' \in \bar{X}$, in general, $\lambda(a, x, \theta)$ is not independent of θ. Moreover, because $\hat{\alpha}(a, x) / \xi(x)$ is independent of $\bar{b}$, the first term on the right-hand side of (26) must be independent of x. For this to be true $u\left(a, \bar{b}(a)(\theta), \theta\right)$ must be independent of θ and $\sum_{\theta \in \Theta} \lambda(a, x, \theta) / \xi(x) := \varphi(a)$ independent of x. Moreover, because the first term on the right-hand side of (26) is independent of x, $\hat{\alpha}(a, x) / \xi(x)$ must also be independent of x. Define $v(a) = \hat{\alpha}(a, x) / \xi(x)$. By definition, $\bar{b}$ is the unique element in its equivalence class that has the property that $u\left(a, \bar{b}(a)(\theta), \theta\right)$ is independent of θ. Define $u\left(a, \bar{b}(a)(\theta), \theta\right) = u\left(a, \bar{b}(a)\right)$ for all $\theta \in \Theta$. Hence, $V\left(a, \bar{b}(a), x\right)$ is independent of x, thus $V\left(a, \bar{b}(a), x\right) = \varphi(a) u\left(a, \bar{b}(a)\right) + v(a) = U\left(\bar{b}(a), a\right)$ and, consequently, $\kappa(a) = 1$.

Thus

$$U\left(a,\bar{b}\left(a\right)\right)=\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\xi\left(x\right)}u\left(a,\bar{b}\left(a\right)\left(\theta\right);\theta\right)+\frac{\hat{\alpha}\left(a,x\right)}{\xi\left(x\right)}. \tag{27}$$

This completes the proof of Lemma 11. ♣

Define $\widetilde{u}\left(a,b\left(\theta\right);\theta\right)=\left[\varphi\left(a\right)u\left(a,b\left(\theta\right);\theta\right)+v\left(a\right)\right]$ then, by equation (27),

$$U\left(a,\bar{b}\left(a\right)\right)=\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\xi\left(x\right)\varphi\left(a\right)}\left[\varphi\left(a\right)u\left(a,\bar{b}\left(a\right)\left(\theta\right);\theta\right)+v\left(a\right)\right]=\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\xi\left(x\right)}\widetilde{u}\left(a,\bar{b}\left(a\right)\left(\theta\right);\theta\right). \tag{28}$$

But, by Lemma 11, $\sum_{\theta\in\Theta}\lambda\left(a,x,\theta\right)=\xi\left(x\right)\varphi\left(a\right)$. Hence the representation (7)implies

$$I\mapsto\sum_{x\in\bar{X}}\left[\sum_{\theta\in\Theta}\frac{\lambda\left(a_{I(x)},x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}\widetilde{u}\left(a_{I(x)},\bar{b}\left(a_{I(x)}\right)\left(\theta\right);\theta\right)\right]. \tag{29}$$

For every $(a,b)\in A\times B$ let $\bar{b}\in\mathcal{B}\left(\succcurlyeq\right)$ be such that $(a,b)\sim^{x}\left(a,\bar{b}\left(a\right)\right)$ for all $x\in\bar{X}$. Then, by the inclusively of $\mathcal{B}$,

$$\sum_{\theta\in\Theta}\frac{\lambda\left(a_{I(x)},x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}\widetilde{u}\left(a_{I(x)},\bar{b}_{I(x)}\left(a_{I(x)}\right)\left(\theta\right);\theta\right)=\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}\widetilde{u}\left(a_{I(x)},b_{I(x)}\left(\theta\right);\theta\right). \tag{30}$$

Thus, by the representation (29),

$$I\mapsto\sum_{x\in\bar{X}}\left[\sum_{\theta\in\Theta}\frac{\lambda\left(a,x,\theta\right)}{\sum_{\theta\in\Theta}\lambda\left(a_{I(x)},x,\theta\right)}\widetilde{u}\left(a_{I(x)},b_{I(x)}\left(\theta\right);\theta\right)\right].$$

For all $x\in X,a\in A$ and $\theta\in\Theta$, define the joint subjective probability distribution on $\Theta\times\bar{X}$ by

$$\pi\left(x,\theta\mid a\right)=\frac{\lambda\left(a,x,\theta\right)}{\sum_{x'\in\bar{X}}\sum_{\theta'\in\Theta}\lambda\left(a,x',\theta'\right)}. \tag{31}$$

Since $\sum_{\theta\in\Theta}\lambda(a,x,\theta)=\xi(x)\varphi(a)$, for all $x\in\bar{X}$,

$$\sum_{\theta\in\Theta}\pi(x,\theta\mid a)=\frac{\xi(x)\varphi(a)}{\sum_{x'\in\bar{X}}\xi(x')\varphi(a)}=\frac{\xi(x)}{\sum_{x'\in\bar{X}}\xi(x')}. \tag{32}$$

Define the subjective probability of $x\in\bar{X}$ as follows:

$$\mu(x)=\frac{\xi(x)}{\sum_{x'\in\bar{X}}\xi(x')}. \tag{33}$$

Then the subjective probability of x is given by the marginal distribution on X induced by the joint distributions $\pi(\cdot,\cdot\mid a)$ on $X\times\Theta$ and is independent of a.

Define the subjective posterior on Θ distribution by

$$\pi(\theta\mid x,a)=\frac{\pi(x,\theta\mid a)}{\mu(x)}=\frac{\lambda(a,x,\theta)}{\sum_{\theta\in\Theta}\lambda(a,x,\theta)}, \tag{34}$$

and define the subjective prior on Θ by:

$$\pi(\theta\mid o,a)=\frac{\lambda(a,o,\theta)}{\sum_{\theta\in\Theta}\lambda(a,o,\theta)}. \tag{35}$$

Substitute in (29) to obtain the representation (3),

$$I\mapsto\sum_{x\in\bar{X}}\mu(x)\left[\sum_{\theta\in\Theta}\pi\left(\theta\mid x,a_{I(x)}\right)\widetilde{u}\left(a_{I(x),}b_{I(x)}(\theta),\theta\right)\right]. \tag{36}$$

(Necessity) The necessity of (A.1), (A.2) and (A.3) follows from Wakker (1989) Theorem III.4.1. To see the necessity of (A.4), suppose that $I_{-x}(a,b_{-\theta}r)\succcurlyeq I_{-x}\left(a,b'_{-\theta}r'\right)$, $I_{-x}\left(a,b'_{-\theta}r''\right)\succcurlyeq I_{-x}(a,b_{-\theta}r''')$, and $I_{-x'}\left(a',b''_{-\theta}r'\right)\succcurlyeq I_{-x'}\left(a',b'''_{-\theta}r\right)$. For all $(a,b,x)\in A\times B\times\bar{X}$ let $G(a,b,x):=\sum_{\theta'\in\Theta-\{\theta\}}\pi(\theta'\mid x,a)\widetilde{u}(a,b(\theta'),\theta')$ then, by the representation (3),

$$G(a,b,x)+\pi(\theta\mid x,a)\widetilde{u}(a,r,\theta)\geq G(a,b',x)+\pi(\theta\mid x,a)\widetilde{u}(a,r',\theta) \tag{37}$$

$$G(a,b',x)+\pi(\theta\mid x,a)\,\widetilde{u}(a,r'',\theta)\geq G(a,b,x)+\pi(\theta\mid x,a)\,\widetilde{u}(a,r''',\theta) \tag{38}$$

and

$$G(a,b'',x')+\pi(\theta\mid x',a')\,\widetilde{u}(a',r',\theta)\geq G(a',b''',x')+\pi(\theta\mid x',a')\,\widetilde{u}(a',r,\theta)\,. \tag{39}$$

But (37) and (38) imply that

$$\widetilde{u}(a,r'',\theta)-\widetilde{u}(a,r''',\theta)\geq\frac{G(a,b,x)-G(a,b',x)}{\pi(\theta\mid x,a)}\geq\widetilde{u}(a,r',\theta)-\widetilde{u}(a,r,\theta)\,. \tag{40}$$

Inequality (39) implies

$$\widetilde{u}(a',r',\theta)-\widetilde{u}(a',r,\theta)\geq\frac{\sum_{\theta'\in\Theta-\{\theta\}}\pi(\theta'\mid x',a')\left[\widetilde{u}(a',b'''(\theta'),\theta')-\widetilde{u}(a',b''(\theta'),\theta')\right]}{\pi(\theta\mid x',a')} \tag{41}$$

But (40) and (41) imply that

$$\widetilde{u}(a',r'',\theta)-\widetilde{u}(a',r''',\theta)\geq\frac{\sum_{\theta'\in\Theta-\{\theta\}}\pi(\theta'\mid x',a')\left[\widetilde{u}(a',b'''(\theta'),\theta')-\widetilde{u}(a',b''(\theta'),\theta')\right]}{\pi(\theta\mid x',a')}. \tag{42}$$

Hence

$$\sum_{\theta'\in\Theta-\{\theta\}}\pi(\theta'\mid x',a')\left[\widetilde{u}(a',b''(\theta'),\theta')-\widetilde{u}(a',b'''(\theta'),\theta')\right]+\pi(\theta\mid x',a')\left[\widetilde{u}(a',r'',\theta)-\widetilde{u}(a',r''',\theta)\right]\geq 0. \tag{43}$$

Thus, $I_{-x'}\left(a',b''_{-\theta}r''\right)\succcurlyeq I_{-x'}\left(a',b'''_{-\theta}r'''\right)$.

Let $a\in A$, $I\in\mathcal{I}$ *and* $b,b'\in B$, satisfy $I_{-o}(a,b)\sim I_{-o}(a,b')$. Then, by (36),

$$\sum_{\theta\in\Theta}\pi(\theta\mid o,a)\,\widetilde{u}(a,b(\theta),\theta)=\sum_{\theta\in\Theta}\pi(\theta\mid o,a)\,\widetilde{u}(a,b'(\theta),\theta) \tag{44}$$

and, by axiom (A.5) and (36)

$$\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\sum_{\theta\in\Theta}\pi(\theta\mid x,a)\,\widetilde{u}(a,b(\theta),\theta)=\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\sum_{\theta\in\Theta}\pi(\theta\mid x,a)\,\widetilde{u}(a,b'(\theta),\theta). \tag{45}$$

Thus

$$\sum_{\theta\in\Theta}\left[\widetilde{u}(a,b(\theta),\theta)-\widetilde{u}(a,b'(\theta),\theta)\right]\left[\pi(\theta\mid o,a)-\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\pi(\theta\mid x,a)\right]=0. \tag{46}$$

This implies that $\pi(\theta\mid o,a)=\sum_{x\in X}\mu(x)\,\pi(\theta\mid x,a)\,/\,[1-\mu(0)]$.

(If $\pi(\theta\mid o,a)>\sum_{x\in X}\mu(x)\,\pi(\theta\mid x,a)\,/\,[1-\mu(0)]$ for some θ and $\mu(o)\,\pi(\theta'\mid o,a)<\sum_{x\in X}\mu(x)\,\pi(\theta'\mid x,a)\,/\,[1-\mu(0)]$ for some θ', let $\hat{b},\hat{b}'\in B$ be such that $\hat{b}(\theta)>b(\theta)$ and $\hat{b}\left(\hat{\theta}\right)=b\left(\hat{\theta}\right)$ for all $\hat{\theta}\in\Theta-\{\theta\}$, $\hat{b}'(\theta')>b'(\theta')$ and $\hat{b}'\left(\hat{\theta}\right)=b'\left(\hat{\theta}\right)$ for all $\hat{\theta}\in\Theta-\{\theta'\}$ and $I_{-o}\left(a,\hat{b}\right)\sim I_{-o}\left(a,\hat{b}'\right)$. Then

$$\sum_{\theta\in\Theta}\left[\widetilde{u}\left(a,\hat{b}(\theta),\theta\right)-\widetilde{u}\left(a,\hat{b}'(\theta),\theta\right)\right]\left[\pi(\theta\mid o,a)-\sum_{x\in X}\frac{\mu(x)}{1-\mu(0)}\pi(\theta\mid x,a)\right]>0. \tag{47}$$

But this contradicts (A.5)). This completes the proof of (a).

(b) Suppose, by way of negation, that there exist continuous, real-valued function $\hat{u}$ *on* $A\times\mathbb{R}\times\Theta$ and, for every $a\in A$, there is a joint probability measure $\hat{\pi}(\cdot,\cdot\mid a)$ on $\bar{X}\times\Theta$, distinct from those that figure in the representation (3), such that $\succcurlyeq$ on $\mathcal{I}$ is represented by

$$I\mapsto\sum_{x\in\bar{X}}\hat{\mu}(x)\left[\sum_{\theta\in\Theta}\hat{\pi}\left(\theta\mid x,a_{I(x)}\right)\hat{u}\left(a_{I(x)},b_{I(x)}(\theta),\theta\right)\right], \tag{48}$$

where $\hat{\mu}(x)=\sum_{\theta\in\Theta}\hat{\pi}(x,\theta\mid a)$ for all $x\in\bar{X}$, and $\hat{\pi}(\theta\mid x,a)=\hat{\pi}(x,\theta\mid a)/\hat{\mu}(x)$ for all $(\theta,x,a)\in\Theta\times X\times A$.

Define $\kappa(x) = \hat{\mu}(x) / \mu(x)$, for all $x \in \bar{X}$. Then the representation (48) may be written as

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_{I(x)}\right) \tilde{\gamma}\left(\theta, x, a_{I(x)}\right) \kappa(x) \hat{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) \right]. \qquad (49)$$

Hence, by (3), $\hat{u}(a, b(\theta), \theta) = u(a, b(\theta), \theta) / \tilde{\gamma}(\theta, x, a) \kappa(x)$. Thus $\tilde{\gamma}(\theta, x, a) \kappa(x)$ is independent of x. Let $\tilde{\gamma}(\theta, x, a) \kappa(x) = \gamma(\theta, a)$. Then, for $\bar{b} \in \mathcal{B}$,

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi\left(\theta \mid x, a_{I(x)}\right) \frac{u\left(a, \bar{b}(a)\right)}{\gamma(\theta, a)} \right]. \qquad (50)$$

Let $\hat{b} \in B$ be defined by $u\left(a, \hat{b}(\theta), \theta\right) = u\left(a, \bar{b}(a)\right) / \gamma(\theta, a)$ for all $\theta \in \Theta$ and $a \in A$. Then $\hat{b} \sim_a^x \bar{b}(a)$ for all $x \in \bar{X}$, and, by definition 1, $\hat{b} \in \mathcal{B}$. Moreover, if $\gamma(\cdot, \cdot)$ is not a constant function then $\hat{b} \neq \bar{b}$. This contradicts the uniqueness of $\bar{b}$ in definition 1. Thus $\gamma(\theta, a) = \bar{\gamma}$ for all $\theta \in \Theta$ and $a \in A$. But

$$1 = \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \hat{\pi}\left(\theta, x \mid a_{I(x)}\right) = \bar{\gamma} \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta, x \mid a) = \bar{\gamma}. \qquad (51)$$

Hence, $\hat{\pi}(\theta, x \mid a) = \pi(\theta, x \mid a)$ for all $(\theta, x) \in \Theta \times \bar{X}$ and $a \in A$.

Next consider the uniqueness of the utility functions. Clearly, if $\hat{u}(a, \cdot, \theta) = mu(a, \cdot, \theta) + k$, $m > 0$, for all $a \in A$ and $\theta \in \Theta$, then

$$\sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \hat{u}(a, b(\theta), \theta) = m \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta, x \mid a) u(a, b(\theta), \theta) + k, \qquad (52)$$

and $\{\hat{u}(a, \cdot, \theta) \mid a \in A, \theta \in \Theta\}$ is another utility function that, jointly with $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ represents $\succcurlyeq$.

Suppose that $\hat{u}(a, \cdot, \theta) = m(a, \theta) u(a, \cdot, \theta) + k$, where $m(\cdot, \cdot)$ is not a constant function. Define $\hat{b}(\theta, a)$ by $m(a, \theta) u\left(a, \hat{b}(\theta, a), \theta\right) = u\left(a, \bar{b}(a)\right)$ for all $\theta \in \Theta$ and $a \in A$. That such

$\hat{b}$ exists follows from the exclusivity of $\mathcal{B}$. By definition, $\hat{b}(\cdot, a) \sim_a^x \bar{b}(a)$ for all x and $\hat{b} \neq \bar{b}$. This contradicts the is uniqueness of $\bar{b}$ in definition 1. Hence $m(a, \theta)$ must be a constant function.

Consider next $\hat{u}(a, \cdot, \theta) = mu(a, \cdot, \theta) + k(\theta, a)$, and suppose that $k(\cdot, a)$ is not a constant function. Let $\bar{k}(x, a) = \sum_{\theta \in \Theta} \pi(\theta \mid x, a) k(\theta, a)$. Take $a, a' \in A$ and $\bar{b}, \bar{b}' \in \mathcal{B}$ such that $I_{-x'}\left(a, \bar{b}(a)\right) \sim I'_{-x'}\left(a', \bar{b}(a')\right)$ and $\left[\bar{k}(x, a) - \bar{k}(x, a')\right] \neq 0$ for some strategies I, I' and observation x'. Let

$$J = \sum_{x \in \bar{X} - \{x'\}} \sum_{\theta \in \Theta} \left[\pi\left(\theta, x \mid a_{I(x)}\right) u\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) - \pi\left(\theta \mid x, a_{I'(x)}\right) u\left(a_{I'(x)}, b_{I'(x)}(\theta), \theta\right)\right]$$

and

$$\hat{J} = \sum_{x \in \bar{X} - \{x'\}} \sum_{\theta \in \Theta} \left[\pi\left(\theta, x \mid a_{I(x)}\right) \hat{u}\left(a_{I(x)}, b_{I(x)}(\theta), \theta\right) - \pi\left(\theta \mid x, a_{I'(x)}\right) \hat{u}\left(a_{I'(x)}, b_{I'(x)}(\theta), \theta\right)\right].$$

Then

$$\hat{u}\left(a, \bar{b}(a)\right) - \hat{u}\left(a', \bar{b}(a')\right) + \hat{J} = m\left[u\left(a, \bar{b}(a)\right) - u\left(a', \bar{b}(a')\right) + J\right] + \left[\bar{k}(x, a) - \bar{k}(x, a')\right]. \tag{53}$$

But $u\left(a, \bar{b}(a)\right) - u\left(a', \bar{b}(a')\right) + J = 0$ and, by equation (53) $\hat{u}\left(a, \bar{b}(a)\right) - \hat{u}\left(a', \bar{b}(a')\right) + \hat{J} \neq 0$. Hence $\hat{u}(\cdot, \theta)$ does not represent $\succcurlyeq$. This completes the proof of (b).

(c) Next I show that if $\bar{b} \in B^A$ satisfies $\tilde{u}\left(a, \bar{b}(a)(\theta), \theta\right) = \tilde{u}\left(a, \bar{b}(a)(\theta'), \theta'\right)$ for all $\theta, \theta', \in \Theta$, $a \in A$ then $\bar{b} \in \mathcal{B}$. Let $\tilde{u}\left(a, \bar{b}(a)(\theta), \theta\right) = g\left(a, \bar{b}(a)\right)$. Suppose that representation (3) holds and let $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, such that $I_{-x}\left(a, \bar{b}(a)\right) \sim I'_{-x}\left(a', \bar{b}(a')\right)$, $I'_{-x}\left(a'', \bar{b}(a'')\right) \sim I_{-x}\left(a''', \bar{b}(a''')\right)$ and $I''_{-x'}\left(a', \bar{b}(a')\right) \sim I'''_{-x'}\left(a, \bar{b}(a)\right)$. Then the representation (7) implies that

$$\sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\right) + \mu(x)\, g\left(a, \bar{b}(a)\right) = \sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I'(\hat{x})}, b_{I'(\hat{x})}, \hat{x}\right) + \mu(x)\, g\left(a', \bar{b}(a')\right) \tag{54}$$

$$\sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I'(\hat{x})}, b_{I'(\hat{x})}, \hat{x}\right) + \mu(x)\, g\left(a'', \bar{b}(a'')\right) = \sum_{\hat{x}\in\bar{X}-\{x\}} w\left(a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\right) + \mu(x)\, g\left(a''', \bar{b}(a''')\right) \tag{55}$$

and

$$\sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right) + \mu(x')\, g\left(a', \bar{b}(a')\right) = \sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right) + \mu(x')\, g\left(a, \bar{b}(a)\right). \tag{56}$$

Equations (54) and (55) imply that

$$g\left(a, \bar{b}(a)\right) - g\left(a', \bar{b}(a')\right) = g\left(a'', \bar{b}(a'')\right) - g\left(a''', \bar{b}(a''')\right). \tag{57}$$

Equality (56) implies

$$\frac{\sum_{\hat{x}\in\bar{X}-\{x'\}}\left[w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right) - w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right)\right]}{\mu(x')} = g\left(a, \bar{b}(a)\right) - g\left(a', \bar{b}(a')\right). \tag{58}$$

Thus

$$\sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}\right) + g\left(a''', \bar{b}(a''')\right) = \sum_{\hat{x}\in\bar{X}-\{x'\}} w\left(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}\right) + g\left(a'', \bar{b}(a'')\right) \tag{59}$$

Hence $I''_{-x'}\left(a'', \bar{b}(a'')\right) \sim I'''_{-x'}\left(a''', \bar{b}(a''')\right)$ and $\bar{b} \in \mathcal{B}$.

To show the necessity of (A.5) let $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, by the representation $I_{-o}(a, b) \sim I_{-o}(a, b')$ if and only if

$$\sum_{\theta\in\Theta} \pi(\theta \mid o, a)\, \widetilde{u}(a, b(\theta), \theta) = \sum_{\theta\in\Theta} \pi(\theta \mid o, a)\, \widetilde{u}(a, b'(\theta), \theta). \tag{60}$$

But $\pi(\theta \mid o, a) = \sum_{x \in X} \mu(x) \pi(\theta \mid x, a) / [1 - \mu(0)]$. Thus (60) holds if and only if

$$\sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \widetilde{u}(a, b(\theta), \theta) = \sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \widetilde{u}(a, b'(\theta), \theta). \tag{61}$$

But (61) is valid if and only if $I^{-o}(a, b) \sim I^{-o}(a, b')$.

For all I and x, let $K(I, x) = \sum_{y \in X - \{x\}} \mu(y) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \widetilde{u}\left(a_{I(y)}, b_{I(y)}(\theta), \theta\right)$. To show the necessity of (A.6) Then $I_{-x}\left(a, \bar{b}(a)\right) \succcurlyeq I'_{-x}\left(a', \bar{b}(a')\right)$ if and only if

$$K(I, x) + \widetilde{u}\left(a, \bar{b}(a)\right) \geq K(I', x) + \widetilde{u}\left(a', \bar{b}(a')\right) \tag{62}$$

if and only if

$$K(I, x) + \widetilde{u}\left(a, \bar{b}'(a)\right) \geq K(I', x) + \widetilde{u}\left(a', \bar{b}'(a')\right) \tag{63}$$

if and only if $I_{-x}\left(a, \bar{b}'(a)\right) \succcurlyeq I'_{-x}\left(a', \bar{b}'(a')\right)$. ■

4.2 Proof of Theorem 7

Suppose that $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.1)-(A.7), $\mathcal{B}(\succcurlyeq)$ is inclusive and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$. Let $\bar{b}, \bar{b}', \bar{b}'' \in \mathcal{B}$ and $I, I' \in \mathcal{I}$ satisfy the following conditions: (a) $I_{-x}\left(a, \bar{b}(a)\right) \sim I'_{-x}\left(a', \bar{b}(a')\right)$, (b) $I_{-x}\left(a, \bar{b}''(a)\right) \sim I'_{-x}\left(a', \bar{b}''(a')\right)$, and (c) $I_{-x}\left(a, \bar{b}''(a)\right) \succ I_{-x}\left(a, \bar{b}(a)\right) \succ I'_{-x}\left(a, \bar{b}'(a)\right)$. Then, by Theorem 3, (a) and (b) imply that

$$\widetilde{u}\left(a, \bar{b}(a)\right) - \widetilde{u}\left(a, \bar{b}''(a)\right) = \widetilde{u}\left(a', \bar{b}(a')\right) - \widetilde{u}\left(a', \bar{b}''(a')\right). \tag{64}$$

By Definition 6, $(a,E) \sim_L^x (a',E)$ if and only if $I_{-x}\left(a, \bar{b}^*(a)_E \bar{b}'(a)\right) \sim I'_{-x}\left(a', \bar{b}^*(a')_E \bar{b}'(a')\right)$, where $I_{-x}\left(a, \bar{b}^*(a)\right) \sim I'_{-x}\left(a', \bar{b}^*(a')\right)$, $I_{-x}\left(a, \bar{b}'(a)\right) \sim I'_{-x}\left(a', \bar{b}'(a')\right)$ and $I_{-x}\left(a, \bar{b}^*(a)\right) \succ I'_{-x}\left(a, \bar{b}'(a)\right)$, for $\bar{b}^* \in \{\bar{b}, \bar{b}''\}$. For every strategy I and observation x, let

$$k(I_{-x}) = \sum_{x' \in \bar{X} - \{x\}} \sum_{\theta \in \Theta} \pi\left(x', \theta \mid a_{I(x)}\right) \widetilde{u}\left(a_{I(x')}, b_{I(x')}(\theta), \theta\right).$$

By Theorem 5, and conditions (a) and (b),

$$I_{-x}\left(a, \bar{b}(a)_E \bar{b}'(a)\right) \sim I'_{-x}\left(a', \bar{b}(a')_E \bar{b}'(a')\right) \tag{65}$$

if and only if

$$\begin{aligned} & k(I_{-x}) + \eta_a(E)\widetilde{u}\left(a, \bar{b}(a)\right) + (1-\eta_a(E))\widetilde{u}\left(a, \bar{b}'(a)\right) = \\ & k\left(I'_{-x}\right) + \eta_{a'}(E)\widetilde{u}\left(a', \bar{b}(a')\right) + (1-\eta_{a'}(E))\widetilde{u}\left(a', \bar{b}'(a')\right) \end{aligned} \tag{66}$$

and

$$I_{-x}\left(a, \bar{b}''(a)_E \bar{b}'(a)\right) \sim I'_{-x}\left(a', \bar{b}''(a')_E \bar{b}'(a')\right) \tag{67}$$

if and only if

$$\begin{aligned} & k(I_{-x}) + \eta_a(E)\widetilde{u}\left(a, \bar{b}''(a)\right) + (1-\eta_a(E))\widetilde{u}\left(a, \bar{b}'(a)\right) = \\ & k\left(I'_{-x}\right) + \eta_{a'}(E)\widetilde{u}\left(a', \bar{b}''(a')\right) + (1-\eta_{a'}(E))\widetilde{u}\left(a, \bar{b}'(a')\right). \end{aligned} \tag{68}$$

Hence

$$\eta_a(E)\left[\widetilde{u}\left(a, \bar{b}(a)\right) - \widetilde{u}\left(a, \bar{b}''(a)\right)\right] = \eta_{a'}(E)\left[\widetilde{u}\left(a', \bar{b}(a')\right) - \widetilde{u}\left(a', \bar{b}''(a')\right)\right]. \tag{69}$$

Equation (64) implies that equation (69) holds if and only if $\eta_a(E) = \eta_{a'}(E)$. Thus $(a,E) \sim_L^x (a',E)$ for all $a, a' \in A$ and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ if and only if $\eta_a(E) = \eta_{a'}(E)$. Hence $\succcurlyeq$ on $\mathcal{I}$ satisfies (A.9) if and only if $\eta_a(E) = \eta_{a'}(E)$. ■

4.3 Proof of Theorem 9

Let $\mathcal{A}$ denote the algebra generated by $\left(\wedge_{i=1}^{k}\mathcal{T}_{a_i}\right)\wedge\mathcal{Y}$, the join of the partitions $\mathcal{T}_{a_1},...,\mathcal{T}_{a_k}$, where k is an integer and $\mathcal{Y}$. Elements of $\mathcal{A}$ include all the intersections of sets belonging to distinct partitions $\mathcal{T}_{a_1},...,\mathcal{T}_{a_k}$. Because $\left(S,A,\Theta,\bar{X}\right)$ is rich each event involving such interactions is equal to a "simple" event, $E\in\mathcal{E}_a$, for some $a\in A$. By Theorem 5, the probability of E is unique and is given by $\eta_a\left(E\right)$. Define the probability measure P_0 on $\mathcal{A}$ as follows:

$$P_0\left(E\right)=\eta_a\left(E\right), \tag{70}$$

for all $E\in\mathcal{A}$.

But $\mathcal{E}$ is the $\sigma-$algebra generated by $\mathcal{A}$. Hence, by Billingsley (1986) Theorem 3.1, P_0 has a unique extension, P, to $\mathcal{E}$. Then P is a probability measure on $(S,\mathcal{E})$. In particular, let $P\left(\mathcal{T}_a\left(\theta\right)\mid x\right)=P\left(\{x\}\times\mathcal{T}_a\left(\theta\right)\right)/P\left(\{x\}\times\Omega\right)$, then

$$P\left(\mathcal{T}_a\left(\theta\right)\mid x\right)=\eta_a\left(\mathcal{T}_a\left(\theta\right)\mid x\right)=\pi\left(x,\theta\mid a\right) \tag{71}$$

and

$$P\left(x\right):=P\left(\{x\}\times\Omega\right)=\eta\left(x\right)=\sum_{\theta\in\Theta}\pi\left(x,\theta\mid a\right)=\mu\left(x\right). \tag{72}$$

Substitute $P\left(\mathcal{T}_a\left(\theta\right)\mid x\right)$ for $\pi\left(\theta\mid x,a\right)$ and $P\left(x\right)$ for $\mu\left(x\right)$ in Theorem 3, to obtain the representation (6). ■

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