FOUNDATIONS OF BAYESIAN THEORY

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Abstract

This paper states necessary and sufficient conditions for the existence, uniqueness, and updating according to Bayes' rule, of subjective probabilities representing individuals' beliefs. The approach is preference based, and the result is an axiomatic subjective expected utility model of Bayesian decision making under uncertainty with state-dependent preferences. The theory provides foundations for the existence of prior probabilities representing decision makers' beliefs about the likely realization of events and for the updating of these probabilities according to Bayes' rule.

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1 Introduction

In this paper I study necessary and sufficient conditions for the existence of unique subjective probabilities representing individuals' beliefs and their updating, according to Bayes' rule. The approach is preference based and the result is a Bayesian decision theory. The work is motivated by three shortcomings of subjective expected utility theory. First, and most important, the theory ascribes to decision makers probabilities that do not necessarily represent their beliefs. Second, the theory does not imply the updating of subjective probabilities according to Bayes' rule. Third, the theory requires that preferences be state independent. The alternative theory developed here addresses all this issues.

The search for a choice-theoretic definition of subjective probabilities that represent decision makers' beliefs regarding the likely realization of events began with the pioneering work of Ramsey [24] and de Finetti [4] and attained its definitive formulation in the work of Savage [25]. Ultimately, however, this quest failed to achieve its goal. The definitions of subjective probabilities in these and later works invoke a convention that is neither part of nor implied by the underlying axioms, namely, that the utility functions are state independent. Whereas state-independent preferences are implied by the axiomatic structure, state-independent utility functions are not. In fact, an infinite number of combinations of state-dependent utilities representing the same risk preferences and arbitrary probability measures are consistent with the axioms. Consequently, the curvature of the utility functions (and the ranking of "objective" lotteries, if such lotteries exist, as, for example, in Anscombe and Aumann [1]) must be independent of the underlying states, but the utility functions themselves may be positive linear transformations of each other.[1] In many situations this convention is untenable. Moreover, if a decision maker's valuations of outcomes are not independent of the underlying states, the imposition of state-independent utility functions means that, even when the decision maker's beliefs (that is, a binary relation on the set of events that have the interpretation "more likely of being realized than") are representable by a probability measure, they may be inconsistent with the subjective probabilities ascribed to the decision maker by the theory.

Choices among alternative courses of action, or acts, reveal the decision maker's marginal rates of substitution between outcomes across states.

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[1] See the discussion in Drèze [5]; Schervish, Seidenfeldt, and Kadane [26]; Karni [13], [15], and Karni and Schmeidler [17].

These trade-offs confound subjective probabilities and marginal utilities and are too coarse to allow a meaningful separation of the two. Misconstrued separation of probabilities and utilities may result in inconsistencies between verbal expression of preferences and observed choice behavior.[2]

A different issue concerns the definition of null events. Ideally, an event is designated as null if the decision maker believes it to be impossible. The standard choice-theoretic formalization of this idea involves the preference ranking of acts that agree outside the given event. If the decision maker displays indifference among all such acts, then the event is defined as null and is assigned zero probability. However, by this definition there is no distinction between an event that the decision maker perceives as impossible and one whose possible outcomes he perceives as equally desirable. In fact, it is possible that the theory ascribes zero probability to events that the decision maker believes to be possible (or even likely). Consider, for instance, a passenger about to board a flight and suppose that, being unattached and having no dependents, he is indifferent to the size of his estate in the event that he dies. (Such a person would decline flight insurance regardless of the terms of the policy.) By the customary definition, for such passenger a plane crash is a null event and is assigned zero probability, even though the passenger may believe that the plane could crash. This problem renders the representation of beliefs by subjective probabilities dependent on the implicit and *unverifiable assumption* that in every event some outcomes are strictly more desirable than others.[3] If this assumption is not warranted the procedure may result in misrepresentations of beliefs. The model developed here overcomes this problem.

From the point of view of Bayesian statistics, to which subjective expected utility theory is supposed to provide a choice-theoretic foundation of prior probabilities, the failure to obtain a correct representation of beliefs by probabilities is a fundamental flow. Moreover, with one exception, while subjective expected utility theory is consistent with the updating of the subjective probabilities according to Bayes' rule, it does not imply it.[4] The exception is Ghirardato [8], who replaced Savage's Sure Thing Principle

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[2] See the example and discussion in Karni [15].

[3] See Karni, Schmeidler, and Vind [18].

[4] The notion of conditional preferences on acts is well defined in subjective expected utility theory. These conditional preferences are sometimes interpreted as the updated preferences. However, this interpretation, appealing as it may be, is not implied by the axioms. In other words, the axioms do not imply that if a decision maker receives information that makes him believe that a certain event obtains, he must update his prior probability on this event equiproportionally. A more detailed discussion of this point is provided in Kyburg [20].

by dynamic consistency to obtain an axiomatic subjective expected utility theory that implies the updating of the prior according to Bayes rule (see Section 3.2 below for a more detailed discussion). In view of these observations, it is natural to define subjective expected utility maximizing decision makers as *Bayesian* if, in addition to being subjective expected utility theory maximizers they also update their prior beliefs according to Bayes' rule.

A last, well known, criticism of subjective expected utility theory is its requirement that the preferences be state independent. This requirement imposes severe limitations on its possible applications. For example, the theory is inappropriate for analysis of the demand for health or life insurance.

The failure of the choice-theoretic models to quantify decision makers' beliefs by a probability measure is due to the restrictive nature of preference relations defined solely on acts (that is, on functions from the set of states of nature to the set of consequences). The extension of the choice set to include conditional acts allows the expression of preferences that makes it possible to separate utilities from probabilities in a more satisfactory manner.[5] In view of the role played by preferences on conditional acts in the subsequent analysis, their interpretation merits some elaboration.

Conditional acts may be thought of as alternatives from which the decision maker could chose were he informed that the conditioning event obtains. Such acts may be deliberately invoked by decision makers when trying to clarify to themselves, or articulate to others, how they would choose among alternative courses of action if they acquire new information pertinent to their decision. Savage [25] uses this interpretation to justify his celebrated Sure Thing Principle. To motivate this principle, he gives the following example (italics are mine):

> A businessman contemplates buying a certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, *to clarify the matter to himself*, he asks whether he would buy it if he *knew* that the Republican candidate were going to win, and decides

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[5] Pflanzagl [23], Luce and Krantz [21], Fishburn [7], and Drèze and Rustichini [6] studied preferences on conditional acts. Luce and Krantz [21] maintain that, in many circumstances, decisions delimit which events may obtain and that in such circumstances the application of Savage's theory is cumbersome and unintuitive. They propose instead a theory based on choice among conditional acts that, they believe, is simpler and more natural. Their critical view of the adequacy of Savage's theory is shared by Fishburn, according to whom "although the Luce-Krantz theory might seem a bit more intricate than Savage's, it surely comes closer to making contact with the structure of actual decision situations" (Fishburn [7] p. 5).

> that he would do so. Similarly, he considers whether he would buy if he *knew* that the Democratic candidate would win, and again finds that he would do so. Seeing that he would buy in either event, he decides that he should buy. (Savage [25] p. 21)

The businessman compares the act of buying conditional on the event that the Republican candidate wins and the act of not buying conditional on the same event. He then proceeds to compare the same two acts conditional on the complementary event. Because the decision maker knows the event that obtains, presumably he is indifferent among all acts that agree on that event. Insofar as he is concerned, comparison between two acts conditional on any given event is independent of the values the two acts assume outside the conditioning event. In other words, an act conditional on an event may be regarded as the subset of all the unconditional acts that agree on that event, and preferences among conditional acts may be expressed in terms of preferences among subsets of unconditional acts that agree with the conditional acts on the conditioning event.

I consider a choice set that includes all conditional acts. This means that, in the above example, the businessman can contemplate a choice between buying the property knowing that the Republican candidate were going to win and buying knowing that the Democratic candidate were going to win. If the businessman could choose the next president as well as whether or not to buy the property, the choice set would have obvious meaning. Such choices rarely present themselves, however, and because they involve distinct conditioning events, may not even be possible. Thus preferences over acts conditional on distinct events may be expressed only verbally. This departure from the traditional choice-theoretic approach merits elaboration. Preference relations over conditional acts may be decomposed as follows: First, preferences among acts conditional on the same event, are taken to represent a decision maker's choice behavior were he to learn that the conditioning event obtained.[6] Second, to link the preference relations on acts conditional on distinct events, suppose that, given any unconditional act, the decision maker can identify consequences (that is, state-outcome pairs) among which he would be indifferent to learning which consequence obtains. In the above example, this means that the businessman can contemplate placing a bet on the outcome of the presidential election that will leave him indifferent between the two possible election outcomes. In this instance the same proposition could be stated in the language of choice, namely, if, after placing his bet, the businessman happens to find himself in a position in

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[6] See discussion in Arrow [2] and Ghirardato [8].

which he could cast the deciding vote, he would be indifferent between voting for the Republican or the Democratic candidate. In general, however, such choices are impossible and the indifference can be expressed only verbally. Now suppose that there are unconditional acts, dubbed constant-valuation acts, each representing state-outcomes pairs among which the decision maker is indifferent. For every given constant-valuation act, I assume that all the induced conditional acts are equally preferred.

Subject to this understanding of the meaning of the preference relations considered here, the significance of the analysis that follows is that it identifies necessary and sufficient conditions for the existence and uniqueness of a probability measure representing decision makers prior beliefs and for the updating of these probabilities and beliefs according to Bayes' rule. Moreover, the theory is general in the sense of accommodating state-dependent as well as state-independent preferences. The underlying premise is that the mental processes at work - namely, the assessment of the likelihood of events and the valuation of outcomes - are the same whether or not the preferences are state independent. It is reassuring, therefore, that both cases are addressed using the same approach and that state-independent preferences are merely a special instance of the more general model. As in Savage [25], the probabilities in this model do not enter as primitives, appearing instead as a derived concept.

The next section describes the model and the main results. Further discussion and review of the relevant literature appears in Section 3. The proofs appear in the appendix.

2 Subjective Expected Utility Theory

2.1 The analytical framework

Let $S = \{1, ..., n\}$, $2 \leq n < \infty$, be a set of *states of nature* one and only one of which is the *true* state. Nonempty subsets of S are *events*. Let $\mathcal{S}$ denote the set of all events. When the true state belongs to the event E, we say that E *obtains*. Uncertainty is the lack of knowledge regarding which state is the true state. For each $s \in S$, let X_s be an interval in $\mathbb{R}$ whose elements are *outcomes* (e.g., monetary payoffs) that are feasible in s.[7] *Unconditional acts* are $n-$tuples $\mathbf{x} = (x_1, ..., x_n)$, where $x_s \in X_s$, representing possible courses of

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[7] The assumption that X_s corresponds to an interval in the real line is aimed at simplifying the exposition. More generally, X_s may be taken to be a linear topological separable quotient space whose elements are indifference classes of the preference relation introduced below.

action. The set of all unconditional acts is the product set $\mathbf{X} := X_1 \times ... \times X_n$. Note that the feasible sets of outcomes do not have to be the same across states. This is a departure from Savage's [25] model, in which the set of acts includes all the constant acts and, consequently, requires that the feasible outcomes be the same in every state.

For each s, denote by $(\mathbf{x}^{-s}, y)$ the unconditional act obtained from $\mathbf{x}$ by replacing its $s-th$ coordinate with $y \in X_s$. For each $E \subset S$ denote by $\mathbf{x}_E\mathbf{y}$ the act that coincides with $\mathbf{x}$ on E and with $\mathbf{y}$ on $S - E$ (that is, $(\mathbf{x}_E\mathbf{y})_s = x_s$ if $s \in E$ and $(\mathbf{x}_E\mathbf{y})_s = y_s$ if $s \in S - E$.)

To begin with assume that decision makers have preference relations, $\succcurlyeq$, (that is complete and transitive binary relations) on the set of unconditional acts, where the strict preference relation $\succ$ and the weak preference relations $\sim$ are the asymmetric and symmetric parts of $\succcurlyeq$, respectively.

Ideally, a state is designated as null if the decision maker believes it impossible for it to be the true state. However, the standard choice-theoretic formalization of this idea – namely, that a state $s \in S$ is null if $(\mathbf{x}^{-s}, y) \sim (\mathbf{x}^{-s}, z)$ for all $y, z \in X_s$ – is too coarse to allow a distinction between situations in which the decision maker believes a state to be impossible from situations in which he is indifferent among all the outcomes that are feasible in that state. The extension of the choice set to include conditional acts makes it possible to differentiate between these situations.

Given a preference relation $\succcurlyeq$ on $\mathbf{X}$, a state s is *obviously nonnull* if $(\mathbf{x}^{-s}, y) \succ (\mathbf{x}^{-s}, z)$ for some $y, z \in X_s$. Let $S' \subset S$ denote the set of obviously nonnull states and denote by $\mathcal{E}'$ the power set of S'. Then an event $E \in \mathcal{E}$ is obviously nonnull if it includes a nonnull state. Henceforth I assume that S contains at least two obviously nonnull states.

Given $E \in \mathcal{E}'$ a *conditional act,* $\mathbf{x}_E$, is the generic element of $\mathbf{X}_E := \prod_{s \in E} X_s$. For each $E \in \mathcal{E}'$ assume that $\mathbf{X}_E$ is endowed with the product topology. Let $\mathbb{X}' = \cup_{E \in \mathcal{E}'} \mathbf{X}_E$ denote the set of all acts conditional on obviously nonnull events and assume that decision makers have preference relations on $\mathbb{X}'$ extending their preference relations $\succcurlyeq$ on $\mathbf{X}$. To simplify the notation at the cost of slight abuse I shall denote this extension by $\succcurlyeq$. (I shall return to the interpretation of this preference relation shortly). Using the preference relation on the set of conditional acts it is possible to distinguish between states that a decision maker believes to be impossible and states in which he is indifferent among all feasible outcomes. The idea is to compare acts conditional on distinct nonnull events one of which includes the state s and the other does not. If the decision maker believes that the state s is impossible, it should not make a difference him whether or not the state s is in the event. However, if he is indifferent among all feasible outcomes in

s but believes that s is possible it does make a difference whether s is in the event or not. Formally:

Definition 1: A state $s \in S$ is *null* if $\mathbf{x}_E \succcurlyeq \mathbf{y}_E$ if and only if $\mathbf{x}_E \succcurlyeq \mathbf{y}_{E\cup\{s\}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $E \in \mathcal{E}'$ such that $s \notin E$; otherwise it is *nonnull.* An event E is null if it consists of null states; otherwise it is nonnull.

Denote by $\mathcal{E}$ the set of all nonnull events. Notice that $\mathcal{E}' \subset \mathcal{E}$, that is, every event that consists of obviously-nonnull states is nonnull however an event may be nonnull yet not obviously so. Let $\mathbb{X} = \cup_{E\in\mathcal{E}} \mathbf{X}_E$ denote the set of all acts conditional on nonnull events and assume that it is endowed with the topology whose open sets are the unions of the open sets in the product spaces $\mathbf{X}_E$, $E \in \mathcal{E}$. Assume that decision makers are characterized by preference relations $\succcurlyeq$ on $\mathbb{X}$ extending their preference relations on $\mathbb{X}'$. Note that decision makers are not supposed to have preferences over acts conditional on events they believe impossible. For expositional convenience, I assume provisionally that all the states are either null or obviously nonnull (that is $\mathcal{E} - \mathcal{E}' = \emptyset$). I revisit this issue in Section 2.5.

I assume throughout that $\succcurlyeq$ is a continuous weak order. Formally, $\succcurlyeq$ is a complete and transitive binary relation on $\mathbb{X}$ such that the sets $\{\mathbf{x}_E \in \mathbb{X} \mid \mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}\}$ and $\{\mathbf{x}_E \in \mathbb{X} \mid \mathbf{y}_{E'} \succcurlyeq \mathbf{x}_E\}$ are closed for all $\mathbf{y}_{E'} \in \mathbb{X}$. The strict preference relation $\succ$ and the indifference relation $\sim$ are defined as usual and have the usual interpretation. The interpretation of the preference relation $\succcurlyeq$ requires some explanation. Consider the statement $\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}$. If $E = E'$, then this statement means that if the decision maker knew that the event E obtained he would prefer the act $\mathbf{x}$ to the act $\mathbf{y}$. This proposition has clear operational meaning: if the decision maker were to leave instructions to an agent to act on his behalf when new information became available, he would instruct the agent to choose act $\mathbf{x}$ rather than $\mathbf{y}$ if he learned that E obtained. Designate the restriction of $\succcurlyeq$ to acts conditional on E by the conditional preference relation $\succcurlyeq_E$. Presumably, these conditional preferences capture the decision makers updating of his preferences in view of the new information. However, if $E \neq E'$, the meaning of $\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}$ is not obvious. To render it meaningful, I invoke the concept of constant valuation acts.[8]

Definition 2: An act $\mathbf{x}^* \in \mathbf{X}$ is a *constant-valuation act* if $\mathbf{x}^*_E \sim \mathbf{x}^*_{E'}$ for all $E, E' \in \mathcal{E}$.

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[8] The idea of constant valuation acts is similar to Drèze's [5] notion of "omnipotent" acts. Similar concepts appear in Karni [14] and Skiadas [27].

The indifference relation in Definition 2 may be expressed verbally by a decision maker as follows: "Given $\mathbf{x}^* \in \mathbf{X}$, were I to choose, I would be indifferent between between choosing E and E'." Note that Definition 2 implies that $\mathbf{x}^*_{\{s\}} \sim \mathbf{x}^*_{\{t\}}$ for all nonnull $s, t \in S$. This means that the decision maker is indifferent between the outcome-state pairs (x^*_s, s) and (x^*_t, t). Underlying Definition 2 is the presumption that, ultimately, the consequence of a decision is a outcome-state pair (x^*_s, s) and that observing an event is a partial resolution of uncertainty. With constant valuation acts the ultimate outcome is of the same valued whichever event obtains. The indifference relation that figures in Definition 2 is a form of consequentialism, asserting that the decision maker's sole concerned is the ultimate consequence, disregarding the particular form in which uncertainty is resolved.

Suppose that decision makers are capable of expressing preferences among constant valuation acts conditional on the same event. Then $\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}$ means that $\mathbf{x}^{**}_E \succcurlyeq \mathbf{x}^*_E$, where $\mathbf{x}^{**}_E \sim \mathbf{x}_E$ and $\mathbf{x}^*_E \sim \mathbf{x}^*_{E'} \sim \mathbf{y}_{E'}$. In other words, the comparison between $\mathbf{x}_E$ and $\mathbf{y}_{E'}$ is indirect and $\succcurlyeq$ on $\mathbb{X}$ is the transitive closure of the conditional preference relations $\succcurlyeq_E, E \in \mathcal{E}$ and the preference relation on conditional constant valuation acts.

Constant valuation acts play a crucial role in the analysis that follows and are analogous to constant acts in Savage [25]. However, unlike Savage, who assumes implicitly that *constant acts are constant-valuation acts,* I recognize that the same outcome may be assigned distinct values in different states.

(A.0) Assume that there exist unconditional constant valuation acts $\overline{\mathbf{x}}$ and $\underline{\mathbf{x}}$ such that $\overline{\mathbf{x}} \succ \underline{\mathbf{x}}$ and $\overline{\mathbf{x}} \succcurlyeq \mathbf{x}_E \succcurlyeq \underline{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{X}$ and that the set of constant valuation acts is convex.[9]

A decision maker's prior *beliefs* are represented by a binary relation, $\trianglerighteq$, on $\mathcal{E}$ that has the following interpretation: For all $T, Q \subset S$, $T \trianglerighteq Q$ means that the decision maker considers the event T as at least as likely to obtain as the event Q. Following Ramsey [24], it is now commonplace to infer a decision maker's beliefs from his willingness to bet on different events. However, given that the outcome valuations may be state dependent, care must taken in defining bets.

Definition 3: A binary relation, $\trianglerighteq$, on $\mathcal{E}$ represents a decision maker's *prior beliefs* if, for all constant valuation acts, $\mathbf{x}^{**}, \mathbf{x}^* \in \mathbf{X}$, satisfying $\mathbf{x}^{**} \succ \mathbf{x}^*$, and for all $T, Q \in \mathcal{E}$, $T \trianglerighteq Q$ if $\mathbf{x}^{**}_T \mathbf{x}^* \succcurlyeq \mathbf{x}^{**}_Q \mathbf{x}^*$. A binary rela-

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[9] The connectedness of the outcome space and the continuity of the preference relation imply that there are infinitely many constant valaution acts. Indeed, implicit in the discussion in the preceding paragraph is the presumption that every conditional act has an equivalent constant valuation act.

tion, $\trianglerighteq_E$, on 2^E represents a decision maker's *posterior beliefs* if for all conditional constant valuation acts, $\mathbf{x}_E^{**}, \mathbf{x}_E^{*} \in \mathbf{X}$, satisfying $\mathbf{x}_E^{**} \succ \mathbf{x}_E^{*}$, and for all $T, Q \in 2^E$, $T \trianglerighteq_E Q$ if $(\mathbf{x}_T^{**}\mathbf{x}^*)_E \succcurlyeq \left(\mathbf{x}_Q^{**}\mathbf{x}^*\right)_E$.

The following terminology is used throughout: A function $W : \mathbb{X} \to \mathbb{R}$ is said to be *continuous additive valued* if there exist continuous functions $w_E(\cdot; s) : X_s \to \mathbb{R}$, for all $E \in \mathcal{E}$ and $s \in E$, such that $W(\mathbf{x}_E) = \sum_{s \in E} w_E(x_s; s)$. The functions $w_E(\cdot; s)$ are called *additive-valued functions.* They are said to *represent* $\succcurlyeq$ if $\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'}$ if and only if $\sum_{s \in E} w_E(x_s; s) \geq \sum_{s \in E'} w_{E'}(y_s; s)$ for all $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$, and to be *jointly cardinal* if $\hat{w}_E(\cdot; s) : X_s \to \mathbb{R}$, for all $E \in \mathcal{E}$ and $s \in E$, is another array of additive valued functions representing $\succcurlyeq$ then $\hat{w}_E(\cdot; s) = \beta w_E(\cdot; s) + \gamma_s$, $\beta > 0$. They are *cardinally measurable fully comparable* if, in addition, $\gamma_s = \gamma$ for all s.

2.2 Axioms

Bayesian decision makers update their beliefs regarding the likely realization of events independently of their valuation of outcomes. To capture this idea, I assume that the preference relation reflects the same valuations of outcome-state pairs regardless of the conditioning event. This is formally expressed by the following two axioms:

(A.1) Cardinal Coherence *For all* $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, $E, E' \in \mathcal{E}'$, $s \in E \cap E'$, *and* $a_s, b_s, c_s, d_s \in X_s$, *if* $(\mathbf{x}^{-s}, a_s)_E \succcurlyeq (\mathbf{y}^{-s}, b_s)_E$, $(\mathbf{y}^{-s}, c_s)_E \succcurlyeq (\mathbf{x}^{-s}, d_s)_E$, *and* $(\mathbf{z}^{-s}, b_s)_{E'} \succcurlyeq (\mathbf{w}^{-s}, a_s)_{E'}$ *then* $(\mathbf{z}^{-s}, c_s)_{E'} \succcurlyeq (\mathbf{w}^{-s}, d_s)_{E'}$.

Axiom (A.1) is an adaptation of Wakker's [31] cardinal consistency axiom. (Wakker [31] discusses the earlier literature on the idea underlying cardinal consistency.) To grasp the meaning of this axiom, think of the preferences $(\mathbf{x}^{-s}, a_s)_E \succcurlyeq (\mathbf{y}^{-s}, b_s)_E$ and $(\mathbf{y}^{-s}, c_s)_E \succcurlyeq (\mathbf{x}^{-s}, d_s)_E$ as indicating that, given E, the "intensity" of the preference for c_s over d_s is sufficiently greater than that of a_s over b_s as to reverse the order of preference between the other coordinates of $\mathbf{x}$ and $\mathbf{y}$. Coherence requires that these intensities are not contradicted by the preference relations on acts conditional on another obviously nonnull events.

The next axiom asserts that the marginal rate of substitution between outcomes in any two states is independent of the conditioning event provided, of course, that these states belong to the conditioning event. Formally,

(A.2) Ordinal Coherence *For all* $\mathbf{x} \in \mathbf{X}$, $E \in \mathcal{E}$, $s, t \in E$, *and* $(x'_s, x'_t) \in X_s \times X_t$, $\left((\mathbf{x}^{-s}, x'_s)^{-t}, x'_t\right) \sim \mathbf{x}$ *if and only if* $\left((\mathbf{x}^{-s}, x'_s)^{-t}, x'_t\right)_E \sim \mathbf{x}_E$.

2.3 A preliminary result

The following conditions are well-known. Their role in the analysis that follows is to ensure the separability of preferences on conditional acts. The first condition requires that, for any given event, $E \in \mathcal{E}$, the preference between any two acts conditional on E be independent of outcomes in states to which the two acts assign the same outcomes. This assumption is analogous to Savage's [26] Sure Thing Principle (see Wakker [31]). Formally:

Conditional Coordinate Independence *For all* $E \in \mathcal{E}$, $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $s \in E$ *and* $w, z \in X_s$, $(\mathbf{x}^{-s}, z)_E \succcurlyeq (\mathbf{y}^{-s}, z)_E$ *if and only if* $(\mathbf{x}^{-s}, w)_E \succcurlyeq (\mathbf{y}^{-s}, w)_E$.

The second condition is introduced to deal with events that have only two nonnull states (see Wakker [31] Ch. III).

Hexagon condition *Let* s *and* t *be two nonnull states and* $E = \{s, t\}$. *Then, for all* x_s, y_s, z_s *in* X_s, x_t, y_t, z_t *in* X_t, *and* $\mathbf{x} \in \mathbf{X}$, *if* $\left((\mathbf{x}^{-s}, x_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, x_t\right)_E$ *and* $\left((\mathbf{x}^{-s}, z_s)^{-t}, x_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, x_s)^{-t}, z_t\right)_E$ *then* $\left((\mathbf{x}^{-s}, y_s)^{-t}, z_t\right)_E \sim \left((\mathbf{x}^{-s}, z_s)^{-t}, y_t\right)_E$.

The following Lemma will be used in the proofs of the subsequent theorems.

Lemma 1 *Let there be at least two obviously nonnull states then: If* $\succcurlyeq$ *on* $\mathbb{X}$ *satisfies (A.1) then it satisfies conditional coordinate independence and the hexagon condition.*

2.4 Subjective expected utility representation of state-dependent preferences

The next theorem establishes the main result: there exists a unique subjective probability distribution on the set of states representing the decision maker's prior beliefs; posterior probabilities obtained from the given prior by Bayes' rule; state-dependent utility functions on the respective sets of outcomes representing the decision maker's valuations; and subjective expected utility representations of the decision maker's conditional and unconditional preferences. Implicit in this result is the notion that the decision maker may imagine himself having to choose among act-event pairs being aware that once the event is chosen, the probabilities he assigns to states belonging to it increase equiproportionally.

Theorem 2 *Suppose that there are at least two obviously nonnull states. Then*

a. *The following two conditions are equivalent:*

(i) Assumption (A.0) holds and the relation $\succcurlyeq$ *is a continuous weak-order on* $\mathbb{X}$ *satisfying (A.1) and (A.2).*

(ii) There exists a probability measure π *on* S *and an array of non-constant, continuous functions* $\{u_s : X_s \to \mathbb{R}\}_{s\in S}$ *such that, for all* $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'} \Leftrightarrow \sum_{s\in E} \pi(s \mid E)\, u_s(x_s) \geq \sum_{s\in E'} \pi(s \mid E)\, u_s(y_s),$$

where, for all $B \in \mathcal{E}$, $\pi(s \mid B) = \pi(s) / \sum_{t\in B} \pi(t)$ *is the probability of state* s *conditional on the event* B, *and* $u_s(\alpha_s x) = u_1(x)$ *for all* $s \in S - \{1\}, x \in X_1$ *and some* $\alpha_s > 0$.

b. *The utility functions* $\{u_s\}_{s\in S}$ *are cardinally measurable fully comparable.*

c. π *is unique and* $\pi(s) = 0$ *if and only if* s *is null.*

Theorem 2 implies that, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s\in S} \pi(s)\, u_s(x_s) \geq \sum_{s\in S} \pi(s)\, u_s(y_s),$$

and, for all $\mathbf{x}_E, \mathbf{y}_E \in \mathbf{X}_E$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_E \Leftrightarrow \sum_{s\in E} \pi(s \mid E)\, u_s(x_s) \geq \sum_{s\in E} \pi(s \mid E)\, u_s(y_s).$$

Hence the probability measure π has the interpretation of a Bayesian prior and, since $\mathbf{x}_E \succcurlyeq \mathbf{y}_E$ if and only if $\sum_{s\in E} \pi(s \mid E)\, u_s(x_s) \geq \sum_{s\in E} \pi(s \mid E)\, u_s(y_s)$, $\pi(\cdot \mid E)$ is the posterior probability distribution on E obtained by the application of Bayes rule.

The proof is rather long. The following remarks may serve as a 'road map' to the proof that, in part (a), $(i) \rightarrow (ii)$, which is the hard part. Because the preference relation $\succcurlyeq$ is a continuous weak order satisfying (A.1) there is an additive-valued representations of $\succcurlyeq$ on $\mathbf{X}_E$ and, for all $E, E' \in \mathcal{E}$, the additive-valued functions that figure in the representation are positive affine transformations of one another. Axiom (A.2) serves to fix a uniform scale of the additive-valued functions across events. Finally, the assumed existence of constant valuation acts serve to identify the utility and probability components implicit the additive valued functions, thus allowing the decomposition of the additive-valued functions into a product of state-dependent (but event-independent) utility and conditional probability.

To understand the role of ordinal coherence notice that absent this axiom the updating of the subjective probabilities does not necessarily obey Bayes rule. Formally,

Theorem 3 *Suppose that there are at least two obviously nonnull states. Then assumption (A.0) holds and the relation $\succcurlyeq$ is a continuous weak-order on $\mathbb{X}$ satisfying (A.1) if and only if there exists a unique family of probability measures $\{\pi(\cdot \mid E) \mid E \in \mathcal{E}\}$ on S and an array of non-constant, continuous functions $\{u_s : X_s \rightarrow \mathbb{R}\}_{s \in S}$ such that, for all $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$,*

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'} \Leftrightarrow \sum_{s \in E} \pi\left(s \mid E\right) u_s\left(x_s\right) \geq \sum_{s \in E'} \pi\left(s \mid E\right) u_s\left(y_s\right),$$

where $u_s\left(\alpha_s x\right) = u_1\left(x\right)$ for all $s \in S - \{1\}, x \in X_1$ and some $\alpha_s > 0$. Moreover, the utility functions $\{u_s\}_{s \in S}$ are cardinally measurable fully comparable, and, for all $E \in \mathcal{E}$ and $s \in E$, $\pi\left(s \mid E\right) = 0$ if and only if s is null.

Unlike in Theorem 2, the conditional probabilities in Theorem 3 do not necessarily satisfy $\pi\left(s \mid E\right) = \pi\left(s\right) / \sum_{t \in E} \pi\left(t\right)$ for all $E \in \mathcal{E}$.

2.5 Subjective expected utility representation of state-independent preferences

State-independent preferences are a special case of the theory just presented. To study this case, assume, without essential loss of generality, that the same outcomes are feasible in all states (i.e., $X_1 = ... = X_n = X$). To help keep this in mind, I denote $\mathbf{X}$ by X^n. Intuitively speaking, state-independent preferences require that the "intensity" of the preferences be the same across states. To formalize this idea I invoke the condition of cardinal coordinate independence of Wakker ([31], Ch. IV).

(A.3) Cardinal Coordinate Independence - *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in X^n$, *nonnull* $s, t \in S$, *and* $a, b, c, d \in X$, *if* $(\mathbf{y}^{-s}, b) \succcurlyeq (\mathbf{x}^{-s}, a)$, $(\mathbf{x}^{-s}, c) \succcurlyeq (\mathbf{y}^{-s}, d)$ *and* $(\mathbf{z}^{-t}, a) \succcurlyeq (\mathbf{w}^{-t}, b)$ *then* $(\mathbf{z}^{-t}, c) \succcurlyeq (\mathbf{w}^{-t}, d)$.

The interpretation of cardinal coordinate independence is analogous to that of cardinal coherence. The relations $(\mathbf{y}^{-s}, b) \succcurlyeq (\mathbf{x}^{-s}, a)$ and $(\mathbf{x}^{-s}, c) \succcurlyeq (\mathbf{y}^{-s}, d)$ indicate that the "intensity" of the preference for c over d in state s is sufficiently greater than that of b over a as to reverse the order of preference between the other coordinates of $\mathbf{x}$ and $\mathbf{y}$. State independence requires that these intensities are not contradicted by the preferences between the same outcomes in any other state t.

The next lemma gives necessary and sufficient conditions for the state-dependent utility functions to be affine transformations of one another.

Lemma 4 *Let* $\succcurlyeq$ *be a continuous weak order on* X^n. *Then the following conditions are equivalent:*

(i) $\succcurlyeq$ *satisfies (A.3).*

(ii) *There exist* $u : X \to \mathbb{R}$ *and positive affine or constant functions* $\varphi_s : u(X) \to \mathbb{R}$ *for all* $s \in S$ *such that, for all* $\mathbf{x}, \mathbf{y} \in X^{\cdot n}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s=1}^{n} \varphi_s \circ u(x_s) \geq \sum_{s=1}^{n} \varphi_s \circ u(y_s).$$

The proof of Lemma 4 follows immediately from Wakker's [31] Theorem IV.2.7 and is omitted.[10]

In general, even if the preference relation has an expected utility representation, *state-independence preferences does not imply state-independent utility functions.* However, if the utility functions are not the same across states then, by Lemma 4, they must be positive affine transformations of one another (i.e., for all $s \in S$ and $x \in X_s$, $u_s(x) := \sigma_s u(x) + \xi_s$, where $\sigma_s > 0$). In other words, the dependence of the evaluation of an outcome on the underlying states is quantifiable by the multiplicative coefficients σ_s and the additive constants ξ_s. Note that if φ_s is a constant function, then s is null. The next theorem captures this fact and is analogous to Theorem 2.

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[10] If the assumption $X_s = X_t$ does not hold, then the utility functions of nonnull states are positive affine transformations of one another over the outcomes that are in the intersection of the sets of feasible outcomes.

Theorem 5 *Suppose that there are at least two obviously nonnull states. Then*

a. *The following conditions are equivalent:*

(i) Assumption (A.0) holds and the relation $\succcurlyeq$ *is a continuous weak-order on* $\mathbb{X}$ *satisfying (A.1)–(A.3).*

(ii) There exists a probability measure π *on* S*, a continuous non-constant function* $u : X \rightarrow \mathbb{R}$*, and for all* $s \in S$ *, there are numbers* $\sigma_s > 0$ *and* ξ_s *such that, for every* $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$*,*

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'} \Leftrightarrow \sum_{s \in E} \pi\left(s \mid E\right)\left[\sigma_s u\left(x_s\right) + \xi_s\right] \geq \sum_{s \in E'} \pi\left(s \mid E'\right)\left[\sigma_s u\left(y_s\right) + \xi_s\right],$$

where, for all $B \in \mathcal{E}$, $\pi\left(s \mid B\right) = \pi\left(s\right) / \sum_{t \in B} \pi\left(t\right)$ *is the probability of the state* s *conditional on the event* B*, and* $u_s\left(\alpha_s x\right) = u_1\left(x\right)$ *for all* $s \in S - \{1\}, x \in X_1$ *and some* $\alpha_s > 0$.

b. *The triplet* (u, σ_s, ξ_s) *is unique. (That is, if* $(v,\ \zeta_s, \tau_s)$ *represent the preference relation as in (ii) then* $v = \beta u + \alpha$ *and, for all* $s \in S$, $\zeta_s = \sigma_s / \beta$*, and* $\tau_s = \xi_s - \alpha \zeta_s$*.)*

c. π *is unique and* $\pi\left(s\right) = 0$ *if and only if* s *is null.*

The proof of Theorem 5 is similar to that of Theorem 2 and is outlined in the appendix.

Notice that if constant valuation acts happen to be constant acts then the utility functions are state independent. In other words, the utility functions that figure in the representation in Theorem 5 are such that $\sigma_s = \sigma$ and $\xi_s = \xi$ for all $s \in S$.

The definitions of subjective probabilities in Theorems 2 and 5 represent the decision makers' prior beliefs. Letting the probability of an event E be given by $\pi\left(E\right) = \sum_{s \in E} \pi\left(s\right)$, the representations in Theorems 2 and 5 and Definition 3 imply that, for all $T, Q \subset S$,

$$T \trianglerighteq Q \quad \Leftrightarrow \quad \pi\left(T\right) \geq \pi\left(Q\right).$$

Moreover, for every given event $E \in \mathcal{E}$ the posterior beliefs, $\trianglerighteq_{E,}$ are represented by the conditional probabilities $\pi(\cdot \mid E)$. Notice that, because the state-space is finite, these are not the only representations of the prior and posterior relations $\trianglerighteq$ and $\trianglerighteq_E$ on $\mathcal{E}$, by probabilities. However, these are the only such probabilities that are compatible with the decision maker's preferences.

2.6 Neither null nor obviously nonnull states

Suppose that there are states that are neither null nor obviously nonnull (that is, states that the decision maker believes are possible but in which all the feasible outcomes are equally preferred). The analysis above may be extended to cover this possibility using the following procedure: Denote the set of all states that are neither null nor obviously nonnull by K and, disregarding the states in K, construct a representation as above to obtain the conditional probabilities on the event $T = S - K$. Then extend the representation to S as follows: Let $t \in K$ and suppose that there are $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ such that $\mathbf{x}_T \nsim \mathbf{y}_T$ and $\mathbf{x}_T \sim \mathbf{y}_{T\cup\{t\}}$. Define $w_{T\cup\{t\}}(y_t;t) = \sum_{s\in T}\left[\pi(s \mid T)\, u_s(x_s) - \pi(s \mid T)\, u_s(y_s)\right].$ Because t in not obviously nonnull, by definition, $\mathbf{y}_{T\cup\{t\}} \sim \left(\mathbf{y}^{-t}, z\right)_{T\cup\{t\}}$ for all $z \in X_t$. Hence, by transitivity, $w_{T\cup\{t\}}(y_t;t)$ is a constant function. Denote its value by w_t. Repeat this procedure for every $t \in K$ that satisfies the condition above.

Given $t \in K$, let $\mathbf{y}^*$ be a constant valuation act satisfying $\mathbf{y}^*_{T\cup\{t\}} \sim \mathbf{y}_{T\cup\{t\}}$. Define $u_t = u_s(y^*_s)$, $s \in T$ (because $\mathbf{y}^*$ is a constant valuation act $u_s(y^*_s)$ is independent of s) and let $p(t) = w_t/u_t$. Repeat this procedure for every state in K using, if necessary, different constant valuation acts. Let $p(K) = \sum_{t\in K} p(t)$ and, for each $s \in S - K$, let $p(s) = \pi(s)(1 - p(K))$. Then the representation theorems 2, 3, and 5 hold with the probability measure $p = (p(1), ..., p(n))$ replacing π. Moreover, p is a unique representation of the decision maker's beliefs, and $p(s) = 0$ if and only if s is null.

3 Discussion

3.1 Beliefs and probabilities

The crucial aspect of the definition of subjective probabilities in this paper is that the probabilities quantify the decision makers' prior and posterior beliefs *correctly*. It is important to note that if one is not interested in the correct representation of beliefs by probabilities (that is, the correct separation of utilities and probabilities), the only meaning of subjective expected utility theory is that it yields separately additive representation. However, separately additive representations may be obtained with less restrictive assumptions. The Anscombe and Aumann [1] model without state-independence, for example, yields separately additive representation (see also the discussion in section 3.3 below). In Karni [15] I argued that a correct representation of beliefs is mandatory if a decision maker's choice behavior is to be consistent with the verbal expressions of his preferences. I show next that correct representation of beliefs is also important for normative economic analysis.

Harsanyi's [11] aggregation theorem shows that if individuals and social preference relations on social-state lotteries satisfy the axioms of expected utility theory of von Neumann and Morgenstern [29] and the social preference relation satisfies a Pareto indifference condition, then the social preferences may be represented as a linear combination of individual utilities. In Harsanyi's theorem the probabilities of the social-state lotteries are given. If these probabilities are subjective, then Harsanyi's approach suggests that individual utilities and probabilities should be aggregated separately into social utilities and probabilities and then combined to obtain an expected utility representation of social preferences. Unfortunately, such an aggregation is inconsistent with Pareto indifference (see Hylland and Zeckhauser [12] and Mongin [22]).

Gilboa, Samet, and Schmeidler [9] argue, convincingly, that the Pareto condition requiring that, when all members of a society are indifferent between two alternatives the social preferences must also be indifferent, is compelling only when the individual members do not hold contradictory beliefs. In other words, without some qualification, the Pareto condition cannot be used to justify social preference over alternatives about which individual members hold conflicting beliefs. They also show that if the Pareto indifference condition is imposed only when there is agreement among individuals' beliefs, then Pareto indifference implies that the social preferences are represented by a subjective expected utility functional with probabilities that are an affine combination of the individual subjective probabilities and a social utility function that is a linear combination of the individual utilities.

Gilboa et al. do not distinguish between probabilities and beliefs. In fact, following the traditional practice in decision theory, they tacitly define beliefs by probabilities and use these probabilities in the formulation of their restricted Pareto condition. This approach opens a gap between their verbal argument, which is stated, quite compellingly, using the language of beliefs, and their formal argument, which is presented in terms of Savage-type ascribed probabilities. What happens if beliefs are not represented by the ascribed probabilities? Not surprisingly, this may lead to two types of errors. Errors of the first type occur when the restricted Pareto condition is not used to justify social preferences when it should be. Errors of the second type occur when the restricted Pareto condition is used to justify social preferences when it should not be.

To grasp the problem, consider the following example. Let there be two individuals, a and b, and two states of nature, 1 and 2. Suppose that individual tastes are captured by state-dependent utility functions defined on the level of wealth as follows:

$\frac{\textit{State}}{\textit{Individual}}$	1	2
a	w^{α}	$2w^{\alpha}$
b	w^{β}	w^{β}

Consider next the beliefs of the individuals.

Case 1: Both individuals believe that state 1 is twice as likely to obtain as state 2. Being subjective expected utility maximizers, their subjective probabilities are $\pi(1) = 2/3$ and $\pi(2) = 1/3$. However, according to traditional subjective expected utility theory, the representation of the individual preferences are:

$$U^{a}((w_1, w_2)) = \frac{1}{2}w_1^{\alpha} + \frac{1}{2}w_2^{\alpha}$$

$$U^{b}((w_1, w_2)) = \frac{2}{3}w_1^{\beta} + \frac{1}{3}w_2^{\beta}$$

Thus since the two individuals appear to disagree on the probabilities, the restricted Pareto condition of Gilboa et. al. does not apply, even though, by their normative argument, it should.

Case 2: Individual a believes that state 1 is four times more likely to obtain than state 2 (i.e., $\pi^{a}(1) = 4/5$ and $\pi^{a}(2) = 1/5$), while individual b believes, as before, that state 1 is twice is likely to obtain as state 2. The preference of the two individuals are represented by:

$$U^{a}((w_1, w_2)) = \frac{2}{3}w_1^{\alpha} + \frac{1}{3}w_2^{\alpha},$$

and

$$U^{b}((w_1, w_2)) = \frac{2}{3}w_1^{\beta} + \frac{1}{3}w_2^{\beta}.$$

In this case, the model of Gilboa et al. implies that restricted Pareto indifference should apply, even though the individuals hold conflicting beliefs. In other words, Gilboa et al. would use the restricted Pareto condition to justify social preferences even though, by their own argument, the situation does not warrant it. *In conclusion, to avoid making errors in using the restricted Pareto condition to justify social preferences, it is necessary to use the corrected probability representations of individual beliefs.*

3.2 Related literature

Karni and Schmeidler (1980)[11] and Karni, Schmeidler, and Vind [18] model subjective expected utility with state-dependent preferences using hypothetical preference relation on hypothetical acts that is linked axiomatically to the preference relation on actual acts in the framework of Anscombe and Aumann [1]. Subsequently, Wakker [30] extended the work of Karni, Schmeidler, and Vind by replacing the roulette lotteries of Anscombe and Aumann with topologically connected consequence spaces. In both cases the consistency requirement imposes state-wise agreement between the hypothetical and the actual conditional preferences on the ranking of lotteries. Karni and Mongin [19] observed that only the model of Karni and Schmeidler (1980) leads to a definition of subjective probabilities that faithfully represents the decision maker's beliefs. Other models, including Karni, Schmeidler, and Vind [18] and Wakker [30], involve a choice of hypothetical probabilities over the states that renders the resulting subjective probabilities arbitrary.[12] A common feature of all these contributions is the reliance on expressed preferences among hypothetical lotteries or objective probability distributions on the states. The model presented here is different in that it relies on preferences on conditional acts. It thus circumvents the need to use probabilities as primitive concepts. In addition, the use of preferences on conditional acts eliminates the last vestige of ambiguity in the definition of the subjective probabilities associated with the possible existence of states in which the decision maker is indifferent among all the outcomes.

Skiadas [28], [29] axiomatized subjective probabilities representing decision makers' beliefs in a model that accommodates state-dependent preferences and admits nonseparability (across states) of the evaluation of acts. In Skiadas' model acts and states are primitive concepts, and preferences are defined on act-events pairs. For any such pair the consequences (utilities) are the decision maker's expression of his holistic valuation act not knowing whether the event occurred. In other words, preferences are an expression of anticipated feeling. Thus, unlike the approach taken here, Skiadas' approach is nonconsequentialist. To link preferences on acts given distinct events, Skiadas uses constant valuation acts. As in this paper this enables him to identify the utility of act across events and, consequently, to derive a subjective probability representation of decision makers' beliefs. The an-

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[11] "An Expected Utility Theory for State-Dependent Preferences," Working Paper 48-80, Foerder Institute for Economic Research, Tel Aviv University.

[12] Grant and Karni [11] extended the work of Karni and Schmeidler (1980) to nonexpected utility preferences. Karni [17] extended it to the framework of Wakker [31].

alytical framework in this paper differs from that of Skiadas as does the axiomatic structure. In particular, Skiadas defines conditional preferences but his model does not include preferences on conditional acts. Hence, unlike the present work and that of Ghirardato described below, his axiomatic structure does not establish a formal link (that is, dynamic consistency) between the preferences on unconditional and conditional acts. Thus, as usual in subjective expected utility theory, the interpretation of the Bayesian updating as reduced-form ex-post Bayesian updating, where a dynamic consistency assumption is implicit. Moreover, Skiadas take the set of null state to be a primitive independent of the preferences. He does not deal with the issue presented by the possible existence of states, or events, that are neither null nor obviously nonnull.

A model of subjective expected utility with Bayesian updating appears in Ghirardato [8], who uses derived conditional preferences over unconditional acts. The framework is similar to that of Savage [26], with the Sure Thing Principle replaced by dynamic consistency, which connects the unconditional and conditional preferences. In addition, the model imposes consistency between unconditional and conditional preferences over constant acts, which together with Savage's P4 imply state-independent preferences. Ghirardato's model implies the existence of a unique prior and event-dependent posterior probability distributions connected through Bayes' rule. However, these probabilities do not necessarily represent the decision makers' beliefs, and the model does not admit state-dependent preferences.

APPENDIX

A. Proof of Lemma 1. Suppose that, for some $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $E \in \mathcal{E}'$, $s \in E$, and $z, w \in X_s$, $(\mathbf{x}^{-s}, z)_E \succcurlyeq (\mathbf{y}^{-s}, z)_E$ and $(\mathbf{y}^{-s}, w)_E \succ (\mathbf{x}^{-s}, w)_E$. In (A.1) let $a_s = b_s = z$ and $c_s = d_s = w$, $E = E'$, $\mathbf{x} = \mathbf{z}$ and $\mathbf{y} = \mathbf{w}$. Then (A.1) implies that $(\mathbf{x}^{-s}, w)_E \succcurlyeq (\mathbf{y}^{-s}, w)_E$, a contradiction. Thus $(\mathbf{x}^{-s}, z)_E \succcurlyeq (\mathbf{y}^{-s}, z)_E$ if and only if $(\mathbf{x}^{-s}, w)_E \succcurlyeq (\mathbf{y}^{-s}, w)_E$. This complete the proof that $\succcurlyeq$ satisfies conditional coordinate independence.

To prove that $\succcurlyeq$ satisfies the Hexagon condition let $E = \{s, t\}$, where s and t are obviously nonnull states. Suppose that $\left((\mathbf{x}^{-s}, x_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, x_t\right)_E$ and $\left((\mathbf{x}^{-s}, z_s)^{-t}, x_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, x_s)^{-t}, z_t\right)_E$. Apply (A.1) with $E = E'$, $a_s = c_s = y_s$, $b_s = x_s$, $d_s = z_s$, $\mathbf{z}^{-s} = \left(\mathbf{x}^{-t}, z_t\right)^{-s}$, $\mathbf{y}^{-s} = (\left(\mathbf{x}^{-t}, y_t\right))^{-s}$, and $\mathbf{w}^{-s} = \mathbf{y}^{-s}$. Then $\left((\mathbf{x}^{-s}, x_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, x_t\right)_E$ is equivalent to $(\mathbf{y}^{-s}, b_s) \sim (\mathbf{x}^{-s}, a_s)$, $\left((\mathbf{x}^{-s}, z_s)^{-t}, x_t\right)_E \sim \left((\mathbf{x}^{-s}, y_s)^{-t}, y_t\right)_E$ is equivalent to $(\mathbf{x}^{-s}, d_s) \sim (\mathbf{y}^{-s}, c_s)$, and $\left((\mathbf{x}^{-s}, y_s)^{-t}, y_t\right)_E \sim \left((\mathbf{x}^{-s}, x_s)^{-t}, z_t\right)_E$ is equivalent to $(\mathbf{y}^{-s}, c_s) := (\mathbf{w}^{-s}, a_s) \sim (\mathbf{z}^{-s}, b_s)$. Apply (A.1) twice to obtain $(\mathbf{z}^{-s}, c_s) \sim (\mathbf{w}^{-s}, d_s)$. Hence $\left((\mathbf{x}^{-s}, y_s)^{-t}, z_t\right)_E \sim \left((\mathbf{x}^{-s}, z_s)^{-t}, y_t\right)_E$. □

B. Proof of Theorem 2 **-** $(a.i) \Rightarrow (\mathrm{a}.ii)$. Let $\bar{\mathbf{x}}$ and $\underline{\mathbf{x}}$ be constant valuation acts such that $\bar{\mathbf{x}} \succcurlyeq \mathbf{x}_E \succcurlyeq \underline{\mathbf{x}}$ for every $\mathbf{x}_E \in \mathbf{X}$ and $\bar{\mathbf{x}} \succ \underline{\mathbf{x}}$. (That such $\bar{\mathbf{x}}$ and $\underline{\mathbf{x}}$ exist is an implication of (A.0)). The connectedness of the outcome space and the continuity of the preference relation imply that for every $\mathbf{x}_E \in \mathbb{X}$ there is a constant valuation act $\mathbf{x}^*$ satisfying $\mathbf{x}^* \sim \mathbf{x}_E$. Note that null states do not affect the preferences among acts. Thus, without loss of generality, when writing $\mathbf{x}_E$ it is assumed that all states in E are nonnull.

Since $\succcurlyeq$ is a continuous weak order satisfying (A.1), by Lemma 1 it also satisfies conditional coordinate independence and the hexagon condition. Hence, by Theorem III.4.1 of Wakker [31], for every event E containing at least two states, there exist additive value functions $W_E : \mathbf{X}_E \rightarrow \mathbb{R}$ that represent $\succcurlyeq$ on $\mathbf{X}_E$ with jointly cardinal continuous functions $\{w_E(\cdot; s) : X_s \rightarrow \mathbb{R}\}_{s \in E}$ (that is, for all $E \in \mathcal{E}_2$, where $\mathcal{E}_2 \subset \mathcal{E}$ denotes the set of events in $\mathcal{E}$ containing at least two nonnull states, and $\mathbf{x}_E, \mathbf{y}_E \in \mathbf{X}_E, \mathbf{x}_E \succcurlyeq \mathbf{y}_E$ if and only if $W_E(\mathbf{x}_E) = \sum_{s \in E} w_E(x_s; s) \geq \sum_{s \in E} w_E(y_s; s) = W_E(\mathbf{y}_E)$). If E is a singleton, say $\{s\}$, then the fact that $\succcurlyeq$ is a continuous weak order imply that there exist a continuous real-valued function $w_{\{s\}}(x_s; s)$ representing $\succcurlyeq$ on X_s (Debreu [3] Theorem I).

Invoking the uniqueness property of the jointly cardinal representation normalize $\{w_E(\cdot; s)\}_{s\in S}$ as follows: For all $E \in \mathcal{E}$ and $s \in E$ set $w_E(\underline{x}_s; s) = 0$ and

$$W_E(\bar{\mathbf{x}}) = \sum_{s\in E} w_E(\bar{x}_s; s) = 1. \tag{1}$$

Next I show that, for all $E \in \mathcal{E}$ and $s \in E$, $w_E(\cdot; s)$ is a positive affine transformation of $w_S(\cdot; s)$.

Claim 1: For all $E \in \mathcal{E}$, $s \in E$, and $x'_s, x_s \in X_s$ $w_E(x'_s; s) \geq w_E(x_s; s)$ if and only if $w_S(x'_s; s) \geq w_S(x_s; s)$.

Proof of Claim 1: Let $(\mathbf{x}^{-s}, x'_s) \succcurlyeq (\mathbf{x}^{-s}, x_s)$. But $(\mathbf{x}^{-s}, x'_s) \succcurlyeq (\mathbf{x}^{-s}, x'_s)$, $(\mathbf{x}^{-s}, x'_s) \succcurlyeq (\mathbf{x}^{-s}, x_s)$ and $(\mathbf{x}^{-s}, x'_s)_E \succcurlyeq (\mathbf{x}^{-s}, x'_s)_E$. Thus, by (A.1), $(\mathbf{x}^{-s}, x'_s)_E \succcurlyeq (\mathbf{x}^{-s}, x_s)_E$. By the same argument $(\mathbf{x}^{-s}, x'_s)_E \succcurlyeq (\mathbf{x}^{-s}, x_s)_E$ implies $(\mathbf{x}^{-s}, x'_s) \succcurlyeq (\mathbf{x}^{-s}, x_s)$. The conclusion is implied by the representation of $\succcurlyeq$ on $\mathbf{X}$ and $\mathbf{X}_E$, respectively. $\diamondsuit$

The next Lemma shows that, for all $E, E' \in \mathcal{E}$, w_E and $w_{E'}$ are nonnegative affine transformations of one another.

Lemma 6 *Assume that there are at least three nonnull states, then the following conditions are equivalent:*

- **(i)** $\succcurlyeq$ *is a continuous weak-order on* $\mathbb{X}$ *satisfying (A.1) and (A.2).*
- **(ii)** *For every* $E \in \mathcal{E}$ *there exist positive affine or constant function* $\phi_E : \cup_{s\in E} w_S(X_s; s) \to \mathbb{R}$ *such that, for all* $s \in E$, $w_E(\cdot; s) = \phi_E \circ w_S(\cdot; s)$, *where* $\{w_E(\cdot; s) : X_s \to \mathbb{R}\}_{s\in S}$ *constitute a jointly cardinal continuous additive representation of* $\succcurlyeq$ *on* $\mathbf{X}_E$, *for all* $E \in \mathcal{E}$.

Proof of Lemma 6 $(i) \Rightarrow (ii)$. Suppose that $\succcurlyeq$ is a continuous weak order satisfying (A.1) and (A.2). Fix $E \in \mathcal{E}_2$ then, by the representation, for every $t \in E$ there exist $\mathbf{w}, \mathbf{z} \in \mathbf{X}$ such that

$$\sum_{r\in S-\{t\}} [w_S(w_r; r) - w_S(z_r; r)] = \zeta > 0, \tag{2}$$

and $\mathbf{x}_E, \mathbf{y}_E \in \mathbf{X}_E$ satisfying

$$\sum_{r\in E-\{t\}} [w_E(x_r; r) - w_E(y_r; r)] = \varepsilon > 0. \tag{3}$$

By continuity of the additive valued functions $w_E(\cdot; s)$ and the connectedness of the sets X_s, for every $\hat{\zeta} \in [-\zeta, \zeta]$, $\hat{\varepsilon} \in [-\varepsilon, \varepsilon]$, and $t \in E$ there exist $\bar{\mathbf{w}}, \bar{\mathbf{z}} \in \mathbf{X}$ and $\bar{\mathbf{x}}_E, \bar{\mathbf{y}}_E \in \mathbf{X}_E$ such that

$$\sum_{r \in S - \{t\}} [w_S(\bar{w}_r; r) - w_S(\bar{z}_r; r)] = \hat{\zeta} \tag{4}$$

and

$$\sum_{r \in E - \{t\}} [w_E(\bar{x}_r; r) - w_E(\bar{y}_r; r)] = \hat{\varepsilon}. \tag{5}$$

For every $E \in \mathcal{E}_2$ and $s \in E$ define a function $\phi_{(E,s)}$ by $w_E(\cdot; s) = \phi_{(E,s)} \circ w_S(\cdot; s)$. Then $\phi_{(E,s)}$ is continuous. To show that it is positive affine function fix $t \in E$ and let $W_t = w_S(X_t; t)$. Then, by the connectedness of X_t and the continuity of $w_S(\cdot; t)$, W_t is an interval in $\mathbb{R}$. Take $\alpha, \beta, \gamma, \delta \in W_t$ such that $-\zeta \leq \alpha - \beta = \gamma - \delta \leq \zeta$ and $-\varepsilon \leq \phi_{(E,t)}(\alpha) - \phi_{(E,t)}(\beta) \leq \varepsilon$. Let $a_t, b_t, c_t, d_t \in X_t$ satisfy $w_S(a_t; t) = \alpha$, $w_S(b_t; t) = \beta$, $w_S(c_t; t) = \gamma$ and $w_S(d_t; t) = \delta$. Take $\hat{\mathbf{w}}, \hat{\mathbf{z}} \in \mathbf{X}$ such that

$$\sum_{r \in S - \{t\}} [w_S(\hat{w}_r; r) - w_S(\hat{z}_r; r)] = \alpha - \beta. \tag{6}$$

By the representation $(\hat{\mathbf{w}}^{-t}; a_t) \sim (\hat{\mathbf{z}}^{-t}; b_t)$ and $(\hat{\mathbf{w}}^{-t}; c_t) \sim (\hat{\mathbf{z}}^{-t}; d_t)$.

Take $\hat{\mathbf{x}}_E, \hat{\mathbf{y}}_E \in \mathbf{X}_E$ such that

$$\sum_{r \in E - \{t\}} [w_E(\hat{x}_r; r) - w_E(\hat{y}_r; r)] = \phi_{(E,t)}(\alpha) - \phi_{(E,t)}(\beta). \tag{7}$$

Since $w_E(\cdot; t) = \phi_{(E,t)} \circ w_S(\cdot; t)$ this implies $(\hat{\mathbf{x}}^{-t}; a_t)_E \sim (\hat{\mathbf{y}}^{-t}; b_t)_E$. Applying (A.1) twice yields $(\hat{\mathbf{x}}^{-t}; c_t)_E \sim (\hat{\mathbf{y}}^{-t}; d_t)_E$. Thus

$$\phi_{(E,t)}(\gamma) - \phi_{(E,t)}(\delta) = \sum_{r \in E - \{t\}} [w_E(\hat{x}_r; r) - w_E(\hat{y}_r; r)] = \phi_{(E,t)}(\alpha) - \phi_{(E,t)}(\beta). \tag{8}$$

By Wakker [30] Lemma 4.4 this implies that $\phi_{(E,t)}$ is affine. Claim 1 implies that $\phi_{(E,t)}$ is nondecreasing, and the restriction of E to include only nonnull states implies that $\phi_{(E,t)}$ is positive, that is, there are numbers $\beta_{(E,t)} > 0$ and $\alpha_{(E,t)}$ such that $w_E(\cdot; t) = \beta_{(E,t)} w_S(\cdot; t) + \alpha_{(E,t)}$.

Next I show that $\beta_{(E,t)}$ is independent of t.

Claim 2: For all $t, s \in E$, $\beta_{(E,t)} = \beta_{(E,s)} = \beta_E$.

Proof of Claim 2: Let $(x'_s, x'_t) \in X_s \times X_t$ and $\mathbf{x} \in \mathbf{X}$ be as in (A.2). Then, by (A.2) and the representation,

$$w_E\left(x'_s; s\right) - w_E\left(x_s; s\right) = w_E\left(x_t; t\right) - w_E\left(x'_t; t\right) \tag{9}$$

if and only if

$$w_S\left(x'_s; s\right) - w_S\left(x_s; s\right) = w_S\left(x_t; t\right) - w_S\left(x'_t; t\right). \tag{10}$$

But $w_E\left(\cdot; s\right) = \beta_{(E,t)} w_S\left(\cdot; s\right) + \alpha_{(E,t)}$ for all E and $t \in E$. Hence equation (9) implies

$$\beta_{(E,s)}[w_S\left(x'_s; s\right) - w_S\left(x_s; s\right)] = \beta_{(E,t)}\left[w_S\left(x_t; t\right) - w_S\left(x'_t; t\right)\right]. \tag{11}$$

Together equations (10) and (11) imply that $\beta_{(E,s)} = \beta_{(E,t)}$ for all $t, s \in E$. $\Diamond$

Let the states 1 and 2 be nonnull and suppose, by way of negation, that for some constant valuation act $\mathbf{x}^*$,

$$w_{\{1\}}\left(x^*_1; 1\right) = w_S\left(x^*_1, 1\right) / w_S\left(\bar{x}_1; 1\right), \; w_{\{2\}}\left(x^*_2; 2\right) = w_S\left(x^*_2, 2\right) / \theta_2. \tag{12}$$

where $\theta_2 > w_S\left(\bar{x}_2; 2\right)$. Let $E = \{1, 2\}$ then, since E is nonnull, the proof of Lemma 6 implies that there exist $\beta_E > 0$ and $\alpha_{(E,\cdot)}$ such that, for all $x \in X_t$ and $t \in E$,

$$w_E\left(x; t\right) = \beta_E w_S\left(x; t\right) + \alpha_{(E,t)}. \tag{13}$$

Moreover, by Definition 2 and the normalization,

$$w_E\left(\underline{x}_1; 1\right) + w_E\left(\underline{x}_2; 2\right) = \beta_E\left[w_S\left(\underline{x}_1; 1\right) + w_S\left(\underline{x}_2; 2\right)\right] + \sum_{t \in \{1,2\}} \alpha_{(E,t)} = 0, \tag{14}$$

and

$$w_E\left(\bar{x}_1; 1\right) + w_E\left(\bar{x}_2; 2\right) = \beta_E\left[w_S\left(\bar{x}_1; 1\right) + w_S\left(\bar{x}_2; 2\right)\right] + \sum_{t \in \{1,2\}} \alpha_{(E,t)} = 1. \tag{15}$$

Hence $\beta_E = \left[w_S\left(\bar{x}_1; 1\right) + w_S\left(\bar{x}_2; 2\right)\right]^{-1}$ and $\sum_{t \in \{1,2\}} \alpha_{(E,t)} = 0$. By Definition 2, for every constant valuation act, $\mathbf{x}^*$,

$$\beta_E\left[w_S\left(x^*_1; 1\right) + w_S\left(x^*_2; 2\right)\right] = w_{\{1\}}\left(x^*_1; 1\right) = w_{\{2\}}\left(x^*_2; 2\right). \tag{16}$$

But

$$w_S\left(x^*_1; 1\right) = w_S\left(\bar{x}_1; 1\right) w_{\{1\}}\left(x^*_1; 1\right) \tag{17}$$

and

$$w_S\left(x_2^*;2\right)=\theta_2 w_{\{2\}}\left(x_2^*;2\right)>w_{\{2\}}\left(x_2^*;2\right)w_S\left(\bar{x}_2;2\right). \tag{18}$$

Hence, using the fact that $w_{\{1\}}\left(x_1^*;1\right)=w_{\{2\}}\left(x_2^*;2\right)$,

$$\beta_E\left[w_S\left(x_1^*;1\right)+w_S\left(x_2^*;2\right)\right]>\beta_E\left(w_S\left(\bar{x}_1;1\right)+w_S\left(\bar{x}_2;2\right)\right)w_{\{2\}}\left(x_2^*;2\right)=w_{\{2\}}\left(x_2^*;2\right). \tag{19}$$

This contradicts equation (16). Hence, for all s and $x_s\in X_s$, $w_{\{s\}}\left(x_s;s\right)=w_S\left(x_s,s\right)/w_S\left(\bar{x}_s;s\right)$.

$(ii)\Rightarrow(i)$. Let $\{w_E\left(\cdot;s\right):X_s\to\mathbb{R}\}_{s\in S}$ constitute a jointly cardinal continuous additive representation of $\succcurlyeq$ on $\mathbf{X}_E$, for all $E\in\mathcal{E}$. Assume that there exist positive affine transformations $\left(\beta_E>0,\alpha_{(E,s)}\right)$ such that $w_E\left(\cdot;s\right)=\beta_E\circ w_S\left(\cdot;s\right)+\alpha_{(E,s)}$ for all $E\in\mathcal{E}_1$ and $s\in E$. That (ii) implies that $\succcurlyeq$ is a continuous weak order satisfying (A.2) is immediate.

To show that (ii) implies (A.1) suppose that $(\mathbf{x}^{-t},a_t)_E\succcurlyeq(\mathbf{y}^{-t},b_t)_E$, $(\mathbf{y}^{-t},c_t)_E\succcurlyeq(\mathbf{x}^{-t},d_t)_E$ and $\left(\mathbf{z}^{-t},b_t\right)_{E'}\succcurlyeq\left(\mathbf{w}^{-t},a_t\right)_{E'}$. By the representation, $(\mathbf{x}^{-t},a_t)_E\succcurlyeq(\mathbf{y}^{-t},b_t)_E$ if and only if

$$w_E\left(a_t;t\right)+\sum_{s\in E-\{t\}}w_E\left(x_s;s\right)\geq w_E\left(b_t;t\right)+\sum_{s\in E-\{t\}}w_E\left(y_s;s\right) \tag{20}$$

and $(\mathbf{y}^{-t},c_t)_E\succcurlyeq(\mathbf{x}^{-t},d_t)_E$ if and only if

$$w_E\left(d_t;t\right)+\sum_{s\in E-\{t\}}w_E\left(x_s;s\right)\leq w_E\left(c_t;t\right)+\sum_{s\in E-\{t\}}w_E\left(y_s;s\right). \tag{21}$$

Hence

$$w_E\left(b_t;t\right)-w_E\left(a_t;t\right)\leq\sum_{s\in E-\{t\}}\left[w_E\left(x_s;s\right)-w_E\left(y_s;s\right)\right]\leq w_E\left(c_t;t\right)-w_E\left(d_t;t\right). \tag{22}$$

Now $w_E\left(\cdot;t\right)=\beta_E w_S\left(\cdot;t\right)+\alpha_{(E,\cdot)}$ and $w_{E'}\left(\cdot;t\right)=\beta_{E'}w_S\left(\cdot;t\right)+\alpha_{(E',\cdot)}$. Hence inequality (22) implies that

$$w_{E'}\left(b_t;t\right)-w_{E'}\left(a_t;t\right)\leq w_{E'}\left(c_t;t\right)-w_{E'}\left(d_t;t\right). \tag{23}$$

By the representation $\left(\mathbf{z}^{-t},b_t\right)_{E'}\succcurlyeq\left(\mathbf{w}^{-t},a_t\right)_{E'}$ if and only if

$$\sum_{s\in E'-\{t\}}w_{E'}\left(z_s;s\right)+w_{E'}\left(b_t;t\right)\geq\sum_{s\in E'-\{t\}}w_{E'}\left(w_s;s\right)+w_{E'}\left(a_t;t\right). \tag{24}$$

Thus

$$w_{E'}\left(b_t;t\right)-w_{E'}\left(a_t;t\right)\geq\sum_{s\in E'-\{t\}}\left[w_{E'}\left(w_s;s\right)-w_{E'}\left(z_s;s\right)\right]. \tag{25}$$

But inequality (23) implies

$$\sum_{s\in E'-\{t\}} w_{E'}\left(z_s;s\right) + w_{E'}\left(c_t;t\right) \geq \sum_{s\in E'-\{t\}} w_{E'}\left(w_s;s\right) + w_{E'}\left(d_t;t\right). \tag{26}$$

Hence, by the representation, $\left(\mathbf{z}^{-t},c_t\right)_{E'} \succcurlyeq \left(\mathbf{w}^{-t},d_t\right)_{E'}$. This completes the proof of Lemma 6. $\Diamond$

By the representation, for all $E\in\mathcal{E}$, $t\in E$, $\mathbf{x}_E\in \mathbf{X}_E$ and $x,x'\in X_t$,

$$(\mathbf{x}^{-t},x)_E \succcurlyeq (\mathbf{x}^{-t},x')_E \quad\Leftrightarrow\quad w_E\left(x;t\right) \;\geq\; w_E\left(x';t\right). \tag{27}$$

By Lemma 6 and the normalization, for all $E\in\mathcal{E}$, $t\in E$ and $x\in X_t$,

$$w_E\left(x;t\right) = \beta_E w_S\left(x;t\right),\ \beta_E > 0, \tag{28}$$

where $\beta_E = \left[\sum_{s\in E} w_S\left(\bar{x}_s;s\right)\right]^{-1}$. (In particular, for all nonnull $t\in S$, $w_{\{t\}}\left(x;t\right) = \beta_{\{t\}} w_S\left(x;t\right)$, where $\beta_{\{t\}} = w_S\left(\bar{x}_t;t\right)^{-1}$).

For each $s\in S$ let $u_s\left(\cdot\right) = w_{\{s\}}\left(\cdot;s\right)$ and define $\pi\left(s\right) = \beta_{\{s\}}^{-1} = w_S\left(\bar{x}_s;s\right)$, if $\beta_{\{s\}} > 0$ (i.e., if s is nonnull) and $\pi\left(s\right) = 0$ otherwise. (Notice that, by this definition, $\beta_E = \left[\sum_{s\in E}\pi\left(s\right)\right]^{-1}$). Then $w_S\left(x;s\right) = \pi\left(s\right)u_s\left(x\right)$ and, for all $\mathbf{x},\mathbf{y}\in\mathbf{X}$,

$$\mathbf{x}\succcurlyeq\mathbf{y} \quad\Leftrightarrow\quad \sum_{s\in S}\pi\left(s\right)u_s\left(x_s\right) \geq \sum_{s\in S}\pi\left(s\right)u_s\left(y_s\right). \tag{29}$$

To show that the functions $\{w_E\left(\cdot,s\right) \mid s\in S, E\in\mathcal{E}\}$ constitute an additive-valued representation of $\succcurlyeq$ on $\{\mathbf{x}_E\in\mathbb{X} | \bar{\mathbf{x}}\succcurlyeq\mathbf{x}_E\succcurlyeq\underline{\mathbf{x}}\}$ take $\mathbf{x}_E,\mathbf{y}_{E'}\in\mathbb{X}$. Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be constant valuation acts such that $\mathbf{x}_E\sim\mathbf{x}^*$ and $\mathbf{y}_{E'}\sim\mathbf{y}^*$. Then, by the representation,

$$\sum_{t\in E} w_E\left(x_t;t\right) = \sum_{t\in S} w_S\left(x_t^*;t\right) \text{ and } \sum_{t\in E'} w_{E'}\left(y_t;t\right) = \sum_{t\in S} w_S\left(y_t^*;t\right).$$

(By definition $\mathbf{x}_E^*\sim\mathbf{x}^*$ and by transitivity, $\mathbf{x}_E\sim\mathbf{x}_E^*$. Thus $\sum_{t\in E} w_E\left(x_t;t\right) = \sum_{t\in E} w_E\left(x_t^*;t\right) = \sum_{t\in E}\beta_E w_S\left(x_t^*;t\right)$. But $\sum_{t\in E}\beta_E w_S\left(x_t^*;t\right) = \sum_{t\in E}\beta_E\pi\left(t\right)u\left(x_t^*;t\right) = \sum_{t\in S}\pi\left(t\right)u\left(x_t^*;t\right) = \sum_{t\in S} w_S\left(x_t^*;t\right)$. Hence $\sum_{t\in E} w_E\left(x_t;t\right) = \sum_{t\in S} w_S\left(x_t^*;t\right)$.)
But $\mathbf{x}_E\succcurlyeq\mathbf{y}_{E'}$ if and only if $\mathbf{x}^*\succcurlyeq\mathbf{y}^*$. Hence, by the representation,

$$\mathbf{x}_E\succcurlyeq\mathbf{y}_{E'} \;\Leftrightarrow\; \sum_{t\in E} w_E\left(x_t;t\right) \geq \sum_{t\in E'} w_{E'}\left(y_t;t\right). \tag{30}$$

Moreover, by equations (28) and using $w_S(x;s) = \pi(s) u_s(x)$ and $\beta_E = \left[\sum_{s\in E} \pi(s)\right]^{-1}$, for all $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'} \Leftrightarrow \sum_{s\in E} \pi(s \mid E) u_s(x_s) \geq \sum_{s\in E'} \pi\left(s \mid E'\right) u_s(y_s), \tag{31}$$

where $\pi(t \mid E) = b_E \pi(t) = \pi(t) / \sum_{s\in E} \pi(s)$.

Because the set of constant valuation acts is convex, if $\hat{\mathbf{x}}$ is a constant valuation act then there is $\boldsymbol{\alpha} \in \mathbb{R}^n_+$ such that $\alpha_1 = 1$ and $\hat{\mathbf{x}} = (\alpha_1\hat{x}, \alpha_2\hat{x}, ..., \alpha_n\hat{x})$. But, by definition, $\hat{\mathbf{x}}_E \sim \hat{\mathbf{x}}$. Hence the representation (31) implies that $\sum_{s\in E} \pi(s \mid E) u_s(\alpha_s\hat{x}) = \sum_{s\in S} \pi(s) u_s(\alpha_s\hat{x})$ for all E. Thus $u_s(\alpha_s\hat{x}) = u_1(\hat{x})$ for all $s \in S - \{1\}$. This completes the proof that (a.*i*) $\Rightarrow$ (a.*ii*).

(a.*ii*) $\Rightarrow$ (a.*i*). The fact that (*ii*) implies that $\succcurlyeq$ is a continuous weak order satisfying (A.2) is straightforward. To show that (*ii*) implies (A.1) take $E \in \mathcal{E}$, $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$, $a_j, b_j, c_j, d_j \in X_j$ and suppose that $(\mathbf{x}^{-j}, a_j) \succcurlyeq (\mathbf{y}^{-j}, b_j)$, $(\mathbf{y}^{-j}, c_j) \succcurlyeq (\mathbf{x}^{-j}, d_j)$, $\left(\mathbf{z}^{-j}, b_j\right)_E \succcurlyeq \left(\mathbf{w}^{-j}, a_j\right)_E$. By (*ii*) $(\mathbf{x}^{-j}, a_j) \succcurlyeq (\mathbf{y}^{-j}, b_j)$ implies

$$\sum_{s\in S-\{j\}} \pi(s) u_s(x_s) + \pi(j) u_j(a_j) \geq \sum_{s\in S-\{j\}} \pi(s) u_s(y_s) + \pi(j) u_j(b_j), \tag{32}$$

and $(\mathbf{y}^{-j}, c_j) \succcurlyeq (\mathbf{x}^{-j}, d_j)$ implies

$$\sum_{s\in S-\{j\}} \pi(s) u_s(x_s) + \pi(j) u_j(d_j) \leq \sum_{s\in S-\{j\}} \pi(s) u_s(y_s) + \pi(j) u_j(c_j). \tag{33}$$

and $\left(\mathbf{z}^{-j}, b_j\right)_E \succcurlyeq \left(\mathbf{w}^{-j}, a_j\right)_E$ implies

$$\sum_{s\in S-\{j\}} \pi(s \mid E) u_s(z_s) + \pi(j \mid E) u_j(b_j) \geq \sum_{s\in S-\{j\}} \pi(s \mid E) u_s(w_s) + \pi(j \mid E) u_j(a_j). \tag{34}$$

But equations (32) and (33) imply that

$$u_j(c_j) - u_j(d_j) \geq u_j(b_j) - u_j(a_j). \tag{35}$$

Hence equations (34) and (35) imply

$$\sum_{s\in S-\{j\}} \pi(s \mid E) u_s(z_s) + \pi(j \mid E) u_j(c_j) \geq \sum_{r\in S-\{j\}} \pi(s \mid E) u_s(w_s) + \pi(j \mid E) u_j(d_j). \tag{36}$$

Equation (36) and (a.*ii*) imply $\left(\mathbf{z}^{-j}, c_j\right)_E \succcurlyeq \left(\mathbf{w}^{-j}, d_j\right)_E$. Hence (a.*ii*) implies (A.1).

That $u_s(\alpha_s \hat{x}) = u_1(\hat{x})$ for all $s \in S - \{1\}$ implies the convexity of the set of constant valuation acts is immediate. Because $\{u_s\}_{s\in S}$ are non-constant, there are constant valuation acts, $\bar{\mathbf{x}}$ and $\underline{\mathbf{x}}$ such that $\bar{\mathbf{x}} \succ \underline{\mathbf{x}}$. Hence (a.*ii*) implies assumption (A.0).

(b) To prove the uniqueness of $\{u_s\}_{s\in S}$ note that, for all $E \in \mathcal{E}$ and $s \in E$, $w_E(\cdot; s) = u_s(\cdot)\,\pi(s \mid E)$. Hence if $\{\hat{u}_s\}_{s\in S}$ is another array of utility function representing $\succcurlyeq$ in the sense of (a.*ii*) then $\hat{w}_E(\cdot, s) := \hat{u}_s(\cdot)\,\pi(s \mid E)$ for all $E \in \mathcal{E}$ and $s \in E$, is another separately additive representation of $\succcurlyeq$. But $\{w_E(\cdot, s) \mid s \in S, E \in \mathcal{E}\}_{s\in S}$ are jointly cardinal, hence $\hat{u}_s(\cdot) = \beta u_s(\cdot) + \gamma_s$. Applying the representation in (a.*ii*) to constant valuation acts imply that for all $E \in \mathcal{E}$, $\sum_{s\in S}\pi(s)\,\gamma_s = \sum_{s\in E}\pi(s \mid E)\,\gamma_s$. Hence $\gamma_s = \gamma$ for all $s \in S$.

(c) Let π and $\{u_s\}_{s\in S}$ satisfy part (a) of Theorem 2. If $t \in S$ is nonnull then for some $\bar{x}_t, \underline{x}_t \in X_t$, and $\mathbf{x} \in \mathbf{X}$, $\left(\mathbf{x}^{-t}, \bar{x}_t\right)_{\{t\}} \succ \left(\mathbf{x}^{-t}, \underline{x}_t\right)_{\{t\}}$. Hence $u_t(\bar{x}_t) - u_t(\underline{x}_t) > 0$. Moreover, $\left(\mathbf{x}^{-t}, \bar{x}_t\right) \succ \left(\mathbf{x}^{-t}, \underline{x}_t\right)$ implies $\pi(t)\left[u_t(\bar{x}_t) - u_t(\underline{x}_t)\right] > 0$. Thus $\pi(t) > 0$. If t is null then, for all $y, z \in X_t$, $\left(\mathbf{x}^{-t}, y\right) \sim \left(\mathbf{x}^{-t}, z\right)$ implying $\pi(t)\left[u_t(y) - u_t(z)\right] = 0$ for all $y, z \in X_t$. Since, by assumption, all states are either null or obviously nonnull, $u_t(y) - u_t(z) \neq 0$ for some $y, z \in X_t$. Hence $\pi(t) = 0$.

To prove the uniqueness of π suppose, by way of negation, that there exists a probability measure, μ, on S and utility functions $\{\hat{u}_s\}_{s\in S}$ that satisfy the representation in (a.*ii*), and $\mu \neq \pi$. Then there are states $s, t \in S$ such that $\mu(s) > \pi(s)$ and $\pi(t) > \mu(t)$. Note that $\mu(s) > \pi(s)$ and $\pi(t) > \mu(t)$ imply that s and t are nonnull. Let $u_s(\cdot) = c_s \hat{u}_s(\cdot)$, for some $c_s > 0$. Then the representation requires that $\mu(s) = c_s \pi(s)/\bar{c}$ for all $s \in S$, where $\bar{c} = \sum_{t\in S} c_t \pi(t)$. But for constant valuation acts, $\mathbf{x}^*$, $u_s(x_s^*) = u_t(x_t^*)$ and $\hat{u}_s(x_s^*) = \hat{u}_t(x_t^*)$. Since $u_s(\cdot) = c_s \hat{u}_s(\cdot)$ for all nonnull s it follows that $c_s = c_t = \bar{c}$. Thus $\mu(s) = c_s \pi(s)/\bar{c}$ implies $\mu(s) = \pi(s)$. A contradiction. $\square$

C. Proof of Theorem 3 The proof of Theorem 3 follows from that of Theorem 2 upon observing that, absent axiom (A.2), Claim 2 in the proof does not hold. Thus it is possible that, for some $t, s \in E$, $\beta_{(E,t)} \neq \beta_{(E,s)}$.

By Lemma 6 and the normalization, for all $E \in \mathcal{E}$, $s \in E$ and $x \in X_t$,

$$w_E(x; s) = \beta_{(E,s)} w_S(x; s)\,,\ \beta_{(E,s)} > 0.$$

In particular, for all nonnull $s \in S$, $w_{\{s\}}(x; s) = \beta_{(\{s\},s)} w_S(x; s)$, where $\beta_{(\{s\},s)} = w_S(\bar{x}_s; s)^{-1}$.

For each $s \in S$ let $u_s(\cdot) = w_{\{s\}}(\cdot; s)$ and define $\pi(s) = \beta^{-1}_{(\{s\},s)} = w_S(\bar{x}_s; s)$, if $\beta_{(\{s\},s)} > 0$ (i.e., if s is nonnull) and $\pi(s) = 0$ otherwise. Hence $w_S(x; s) = \pi(s) u_s(x)$ and, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if $\sum_{s \in S} \pi(s) u_s(x_s) \geq \sum_{s \in S} \pi(s) u_s(y_s)$.

By the normalization, $\sum_{s \in E} \beta_{(E,s)} \pi(s) = 1$ and $\beta_{(E,s)} \pi(s) \geq 0$ for all s, with strict inequality if s is nonnull. Define a probability measure $\pi(\cdot \mid E)$ on S by $\pi(s \mid E) = \beta_{(E,s)} \pi(s)$, $s \in E$ and $\pi(s \mid E) = 0$ otherwise. Then

$$\sum_{s \in E} w_E(x_s; s) = \sum_{s \in E} \beta_{(E,t)} w_S(x_s; s) = \sum_{s \in E} \beta_{(\{E,s)} \pi(s) u_s(x_s) = \sum_{s \in E} \pi(s \mid E) u_s(x_s; s).$$

The conclusions follow by the same arguments as in the proof of Theorem 2. $\square$

D. Proof of Theorem 5 **-** Theorem 5 follows from Theorem 2 with the following additional specifications: Without loss of generality let 1 be a nonnull state and set $u(x) = u_1(x)$, $x \in X$. Then, by Lemma 4 and Theorem 2, for every $s \in S$, $u_s(x) = \sigma_s u(x) + \xi_s$, where $\sigma_s > 0$. Hence, by Theorem 2, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succcurlyeq \mathbf{y} \quad \Leftrightarrow \quad \sum_{s \in S} \pi(s) \sigma_s u(x_s) \geq \sum_{s \in S} \pi(s) \sigma_s u(y_s). \tag{37}$$

and, for every $\mathbf{x}_E, \mathbf{y}_{E'} \in \mathbb{X}$,

$$\mathbf{x}_E \succcurlyeq \mathbf{y}_{E'} \Leftrightarrow \sum_{s \in E} \pi(s \mid E) \left[\sigma_s u(x_s) + \xi_s\right] \geq \sum_{s \in A} \pi\left(s \mid E'\right) \left[\sigma_s u(y_s) + \xi_s\right]. \tag{38}$$

The proof that (ii) implies (i) is follows immediately from the representation.

The proofs of the part (b) is straight forward. The proof of part (c) follow from the proof of part (c) in Theorem 2. $\square$

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