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A new approach to modeling decision-making under uncertainty

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Abstract This paper presents two axiomatic models of decision making under uncertainty that avoid the use of a state space. The first is a subjective expected utility model with action-dependent subjective probabilities and effect-dependent preferences (the case of effect-independent preferences is obtained as a special instance). The second is a nonexpected utility model involving well-defined families of action-dependent subjective probabilities on effects and a utility representation that is not necessarily linear in these probabilities (a probabilistic sophistication version of this model, with action-dependent subjective probabilities is obtained as a special case).

Keywords: Expected utility with action-dependent subjective probabilities and effect-dependent utility, probabilistically sophisticated choice.

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1 Introduction

The notion of states of nature is a cornerstone of modern theories of decision-making under uncertainty. Introduced by Savage [6], a state of nature formalizes the idea of complete resolution of uncertainty–that is, the assignment of a unique consequence to each conceivable course of action.[1] For example, when betting on the outcome of a horse race the states of nature correspond to the orders in which the horses may cross the finish line (these are the states of nature in Anscombe and Aumann [1]).

Theories of decision-making under uncertainty that invoke the notion of a state space (that is, the set of all states of nature) require that the states be so defined as to render the likely realization of alternative events (that is, subsets of the state space) independent of the decision maker's actions and render the valuation of the consequences independent of the state in which they may obtain. Consequently, if the likely outcomes of a horse race may be affected by actions taken by decision makers (for example, a jockey throwing a race) then the outcomes of the race no longer qualify as states of nature. Similarly, taking out health insurance policy is betting on one's state of health. The uncertainty involved is resolved once the true state of the insured's health becomes known. However, the likely realization of alternative states of health is not independent of the life style (e.g., diet and exercise regimen) adopted by the insured. Moreover, in general, the individual valuation of the indemnity is not independent of the insuree's state of health. Hence the states of the decision maker's health

[1] Formally, a state of nature is a function on the set of courses of action, or acts, to the set of consequences.

do not qualify as states of nature. If the framework of Savage is to be maintained, a different, more abstract, formulation of a state space is called for in which the outcomes of the horse race and the states of the decision maker's health become random variables. However–and here is the rub–in many situations involving decision-making under uncertainty, the state space required to meet these conditions is too abstract and/or too complex to contemplate. Except in special circumstances, the state space does not correspond to an image of the world that decision makers invoke when making decisions under uncertainty.[2]

In Karni [2] I developed axiomatic subjective expected utility theories of decision-making under uncertainty that dispense with the idea of a state space. In this paper I pursue this approach using the Anscombe and Aumann [1] device of roulette lotteries. Taking advantage of the richness of the choice space afforded by the availability of roulette lotteries, I develop axiomatic subjective expected utility models with action-dependent subjective probabilities and effect-dependent utilities that are simple and transparent. Moreover, following Machina and Schmeidler [4], [5], I also develop an axiomatic model of decision-making under uncertainty in which action-dependent subjective probabilities are defined but the utility representation is not necessarily linear in these probabilities. This theory allows for effect-dependent preferences and includes, as a special case, a version of Machina and Schmeidler's probabilistically sophisticated choice model with action-dependent subjective probabilities.

In the next section I lay out the analytical framework. The subjective expected utility model is the subject matter of Section 3. Section 4 includes the more general nonexpected

[2] For additional examples, a more detailed discussion, and references, see Karni [2].

utility theory. Concluding remarks appear in Section 5. Section 6 contains the proofs.

2 The Model

2.1 The analytical framework

Let Θ be a finite set whose elements are *effects.* In the examples cited earlier, effects are possible outcomes of a horse race or the states of a person's health. Let A be an abstract set whose elements, referred to as *actions.* Actions correspond to initiatives that may be taken by a decision maker that, he believes, affect the likely realization of alternative effects (e.g., doping a horse prior to running the race, quitting smoking). Let $Z(\theta)$ be an arbitrary set of *prizes* that are feasible if the effect θ obtains. Denote by $\Delta(Z(\theta))$ the set of all simple probability distributions on $Z(\theta)$.[3] Elements of $\Delta(Z(\theta))$ are referred to as roulette lotteries, or simply lotteries.

Bets are effect-contingent lottery payoffs. Formally, a *bet,* b, is a function on Θ such that $b(\theta) \in \Delta(Z(\theta))$. Denote by B the set of all bets (that is, $B := \Pi_{\theta\in\Theta}\Delta(Z(\theta))$). The *choice set* is the product set $\mathbb{C} := A \times B$ whose generic element, (a, b), is an action-bet pair. Action-bet pairs represent conceivable alternatives among which decision makers may have to choose. The set of *consequences* C consists of prize-effect pairs, that is, $C := \{(z, \theta) \mid z \in Z(\theta), \theta \in \Theta\}$.

[3] A distribution is simple if its support is finite.

A decision maker is characterized by a *preference relation* $\succcurlyeq$ on $\mathbb{C}$ (that is, a binary relation having the usual interpretation, namely, $(a,b) \succcurlyeq (a',b')$ means that (a,b) is at least as desireable as (a',b')). In other words, decision makers are supposed to be able to choose, or express preferences, among action-bet pairs presumably taking into account their beliefs regarding the influence of the actions on the likelihood of alternative effects and, consequently, on the desirability of the corresponding bets, and the intrinsic desirability of the actions. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are defined as usual.

For all $b, b' \in B$ and $\alpha \in [0,1]$, defined $(\alpha b + (1-\alpha) b')(\theta) = \alpha b(\theta) + (1-\alpha) b'(\theta)$, for all $\theta \in \Theta$. Assume that the set B is closed under this convex operation. For $p \in \Delta(Z(\theta))$ I use the notation $b_{-\theta}p$ to denote the bet that result from replacing the $\theta-$coordinate of b with the lottery p.

The analysis below invokes the concept of constant valuation bets of Karni [2]. To grasp the idea of constant valuation bets, suppose that there is a subset, $\hat{A}$, of actions and a subset, B^{cv}, of bets such that, for every given $\bar{b} \in B^{cv}$, the variations in the decision-maker's well-being due to the direct impact of the actions in $\hat{A}$ are compensated by the impact of these actions on the likely realization of the different effects. Once accepted by the decision maker, a constant valuation bet manifests itself by the decision maker's indifference among all the actions in $\hat{A}$. If the constant valuation bets are to be well-defined, there must exist sufficiently many distinct actions in $\hat{A}$ so as to render the coordinates of each constant valuation bet unique in the sense of belonging to the same equivalence class of lotteries. To formalize this idea, let $I(b;a) = \{b' \in B \mid (a,b') \sim (a,b)\}$ and $I(p;\theta,b,a) = \{q \in \Delta(Z(\theta)) \mid (a,b_{-\theta}q) \sim (a,b_{-\theta}p)\}$.

Definition 1: A bet $\bar{b} \in B$ is a *constant valuation bet according to* $\succcurlyeq$ if $(a,\bar{b}) \sim (a',\bar{b})$ for all $a,a' \in \hat{A}$, and $b \in \cap_{a\in\hat{A}} I(\bar{b};a)$ if and only if $b(\theta) \in I(\bar{b}(\theta);\theta,\bar{b},a)$ for all $\theta \in \Theta$ and $a \in \hat{A}$.

To grasp the meaning of this definition, note that distinct actions correspond to distinct beliefs regarding the likely realization of the different effects. Definition 1 requires that the cardinality of the subset $\hat{A}$ is at least that of the set of effects Θ. In addition, as the proof of Theorem 1 below indicates, $\hat{A}$ must include $\mid \Theta \mid$ 'independent' actions, inducing systems of linear equations whose unique solutions are the vectors of utilities, $\left(U\left(\bar{b}(\theta),\theta\right)\right)_{\theta\in\Theta}$, associated with the constant valuation bets, $\bar{b} \in B^{cv}(\succcurlyeq)$. To simplify the exposition, without essential loss of generality, I assume, henceforth, that $\hat{A} = A$.[4] In view of Definition 1 and the assumption that $\hat{A} = A$, if $\bar{b},\bar{b}' \in B^{cv}(\succcurlyeq)$ I shall write $\bar{b} \succcurlyeq \bar{b}'$ instead of $(a,\bar{b}) \succcurlyeq (a,\bar{b}')$. Given $p \in \Delta(Z(\theta))$, I denote by $\overline{b_{-\theta}p}$ the constant valuation bet whose $\theta-$ coordinate is p. Since $\succcurlyeq$ is given, to simplify the notation I refer to constant valuation bets according to $\succcurlyeq$ as constant valuation bets and denote $B^{cv}(\succcurlyeq)$ by B^{cv}.

Given $\succcurlyeq$, an effect $\theta \in \Theta$ is *null given the action* a if $(a,b_{-\theta}p) \sim (a,b_{-\theta}q)$ for all $p,q \in \Delta(Z(\theta))$, and $b \in B$, otherwise it is *nonnull given the action* a. Note that, given $\succcurlyeq$, an

[4] In Karni [2] I explain the procedure by which the analysis may be extended to the case in which $\hat{A}$ is a proper subset of A.

effect may be null under some actions and nonnull under others. Denote by $\Theta(a)$ the subset of effects that are nonnull given a and let $\Theta^c(a)$ be its complement in Θ.

Two effects, θ and θ' are said to be *elementarily linked* if there are actions $a, a' \in A$ such that $\theta, \theta' \in \Theta(a) \cap \Theta(a')$. Two effects are said to be *linked* if there exists a sequence of effects $\theta = \theta_0, ..., \theta_n = \theta'$ such that every θ_j is elementarily linked with θ_{j+1}.

The preference relation $\succcurlyeq$ is said to be *nondegenerate* if the strict preference relation, $\succ$, is nonempty, otherwise the preference relation is *degenerate.* I assume throughout, that the preference relation in nondegenerate, every pair of effects is linked and every action-bet pair has an equivalent constant valuation bet. Formally,

(A.0) The preference relation $\succcurlyeq$ is nondegenerate, every pair of effects is linked and, for all $(a, b) \in \mathbb{C}$, there is $\bar{b} \in B^{cv}$ such that $\left(a, \bar{b}\right) \sim (a, b)$.

2.2 Continuous weak orders

The first two axioms are part of all the models below. These axioms are familiar and require no further explanation.

(A.1) (**Weak order**) $\succcurlyeq$ is complete and transitive.

(A.2) (**Action-wise Archimedean**) For all $a \in A$ and $b, b', b'' \in B$, such that $(a, b) \succ (a, b') \succ (a, b'')$ there exist $\alpha, \beta \in (0,1)$ such that $(a, (\alpha b + (1-\alpha) b')) \succ (a, b') \succ$

$(a, (\beta b + (1-\beta) b''))$.

3 Subjective Expected Utility Theory

In this section I explore alternative subjective expected utility models.

3.1 Effect-dependent preferences

Action-wise independence is the independence axiom of expected utility theory applied to elements of the choice set that having the same action as their first coordinate.

(A.3) (**Action-wise Independence**) For all $a \in A$, $b, b', b'' \in B$ and $\alpha \in (0,1]$, $(a,b) \succcurlyeq (a,b')$ if and only if $(a, (\alpha b + (1-\alpha) b'')) \succcurlyeq (a, (\alpha b' + (1-\alpha) b''))$.

The next axiom requires that, for every given effect, the ranking of lotteries be independent of the action. In other words, conditional on the effects, the risk attitude displayed by the decision maker is independent of his actions. Formally,

(A.4) (**Action-independent risk attitudes)** For all $a, a' \in A$, $b \in B$, $\theta \in \Theta(a; \succcurlyeq) \cap \Theta(a'; \succcurlyeq)$ and $p, q \in \Delta(Z(\theta))$, $(a, (b_{-\theta}p)) \succcurlyeq (a, (b_{-\theta}q))$ if and only if $(a', (b_{-\theta}p)) \succcurlyeq (a', (b_{-\theta}q))$.

The first representation theorem below asserts that if assumption (A.0) holds then the axiomatic structure depicted by (A.1) – (A.4) is necessary and sufficient for the existence of a representation based on subjective expected utility with action-dependent probabilities and effect-dependent utilities. In particular, given any action, preferences among bets are represented by a subjective expected utility functional with effect-dependent utility functions.

Theorem 1 *Let $\succcurlyeq$ be a preference relation on $\mathbb{C}$ and suppose that assumption (A.0) holds, then*

(a) *The following conditions are equivalent:*

(a.i) *$\succcurlyeq$ satisfies (A.1) – (A.4).*

(a.ii) *There exists a family of probability measures $\{\pi(\cdot; a) \mid a \in A\}$ on Θ, a family of effect dependent, continuous, utility functions $\{u(\cdot;\theta) : Z(\theta) \to \mathbb{R} \mid \theta \in \Theta\}$, and a continuous function $f : \mathbb{R} \times A \to \mathbb{R}$, increasing in its first argument, such that, for all $(a,b), (a',b') \in \mathbb{C}$,*

$$(a,b) \succcurlyeq (a',b')$$

if and only if

$$f\left(\sum_{\theta\in\Theta} \pi(\theta;a) \sum_{z\in Z(\theta)} u(z;\theta)\, b(z;\theta), a\right) \geq f\left(\sum_{\theta\in\Theta} \pi(\theta;a') \sum_{z\in Z(\theta)} u(z;\theta)\, b'(z;\theta), a'\right). \quad (1)$$

(b) *$\{v(\cdot;\theta) : Z(\theta) \to \mathbb{R} \mid \theta \in \Theta\}$ is another family of utility functions and g is another continuous function representing the preference relation in the sense of (a.ii) if and only* *if, for all* $\theta \in \Theta$, $v(\cdot,\theta) = \lambda u(\cdot,\theta) + \varsigma(\theta)$, $\lambda > 0$, *and, for all* $a \in A$, $g(\lambda x + \varsigma(a), a) = f(x,a)$, *where* $x \in \{\sum_{\theta\in\Theta} \pi(\theta;a) \sum_{x\in Z(\theta)} u(z;\theta) b(z;\theta) \mid b \in B\}$ *and* $\varsigma(a) = \sum_{\theta\in\Theta} \varsigma(\theta)\pi(\theta;a)$.

(c) *The family of probability measures* $\{\pi(\cdot;a) \mid a \in A\}$ *on* Θ *is unique and* $\pi(\theta;a) = 0$ *if and only if* θ *is null given* a.

The function $f(\cdot,a)$ captures the direct effect of the action on the decision maker's well-being. The indirect effect, due to variations in the likely realization of effects, is captured by the probability measures $\{\pi(\cdot;a)\}_{a\in A}$.

3.2 Effect-independent preferences

Consider next the case of effect-independent risk attitudes, that is, the case in which the ranking of lotteries is the same across effects. This is analogous to the Anscombe and Aumann [1] state-independence, or monotonicity, axiom. Formally,

(A.5) (**Effect independence)** For all $a \in A, b \in B$, $\theta, \theta' \in \Theta(a;\succcurlyeq)$, and $p,q \in \Delta(Z(\theta)) \cap \Delta(Z(\theta'))$, $(a, b_{-\theta}p) \succcurlyeq (a, b_{-\theta}q)$ if and only if $(a, b_{-\theta'}p) \succcurlyeq (a, b_{-\theta'}q)$.

The next theorem establishes that if the preference relation satisfies effect independence in addition to the other axioms, then the utility functions that figures in Theorem 1 have the specific functional form $u(z;\theta) = \sigma(\theta)u(z) + \kappa(\theta)$, $\sigma(\cdot) > 0$.

Theorem 2 *Let $\succcurlyeq$ be a preference relation on $\mathbb{C}$ and suppose that assumption (A.0) holds then,*

(a) *The following conditions are equivalent:*

(a.i) *$\succcurlyeq$ satisfies (A.1) – (A.5).*

(a.ii) *There exists a family of probability measures $\{\pi(\cdot; a) \mid a \in A\}$ on Θ, a continuous, real-valued function, u, on $\cup_{\theta \in \Theta} Z(\theta)$, $\sigma \in \mathbb{R}^{\Theta}_{++}$ and $\kappa \in \mathbb{R}^{\Theta}$, and a continuous, real-valued, functions $f : \mathbb{R} \times A \to \mathbb{R}$, increasing in its first argument, such that, for all $(a, b), (a', b') \in \mathbb{C}$,*

$$(a, b) \succcurlyeq (a', b')$$

if and only if

$$f\left(\sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z} [\sigma(\theta) u(z) + \kappa(\theta)] b(z; \theta), a\right) \geq f\left(\sum_{\theta \in \Theta} \pi(\theta; a') \sum_{z \in Z} [\sigma(\theta) u(z) + \kappa(\theta)] b'(z; \theta), a'\right) \tag{2}$$

(b) *v and g is another set of functions representing the preference relation in the sense of (a.ii) if and only if, for all $\theta \in \Theta$, $v = \lambda u + \varsigma(\theta)$, $\lambda > 0$ and $g(\lambda x + \varsigma(a), a) = f(x, a)$, for all $a \in A$ and $x \in \{\sum_{\theta \in \Theta} \pi(\theta; a) \sum_{\theta \in \Theta} [\sigma(\theta) u(z) + \kappa(\theta)] b(z; \theta) \mid b \in B\}$, where $\varsigma(a) = \sum_{\theta \in \Theta} \varsigma(\theta) \pi(\theta; a)$.*

(c) *The family of probability measures $\{\pi(\cdot; a) \mid a \in A\}$ on Θ is unique and $\pi(\theta; a) = 0$ if and only if θ is null given a.*

One implication of Theorem 2 is that effect-independent preferences (or risk attitudes) do not imply effect-independent utility functions. The utility functions are effect independent if and only if constant bets (that is, bets that are constant functions) are constant utility bets. In other words, if $u(z;\theta) = u(z)$ for all $\theta \in \Theta$.

4 Subjective Probabilities without Expected Utility

4.1 Motivation

Machina and Schmeidler [4], [5] argue, convincingly, that a choice-theoretic definition of subjective probabilities does not require that the preference relation satisfy the axioms of subjective expected utility. Invoking the analytical frameworks of Savage [6] and Anscombe and Aumann [1], respectively, they show that a decision maker may be "probabilistically sophisticated," in the sense that his betting behavior implies the existence of unique subjective probabilities on the state space, his choice among acts is determined by his preferences among the induced distributions on the set of outcomes, and his preferences are representable by a utility function that is not necessarily linear in the probabilities. However, as in expected utility theory, the definition of subjective probabilities in the theory of probabilistically sophisticated choice of Machina and Schmeidler is based on an implicit, unverifiable, assumption of state-independent outcome valuation. Moreover, requiring that acts be fully characterized by their induced distributions on the set of consequences, they implicitly assume that the preferences are state independent.

My next objective is to develop, within the analytical framework of Section 2, a decision theory that yields a definition of action-dependent subjective probabilities on the set of effects without requiring that the representation be linear in the probabilities. Unlike the models of Machina and Schmeidler, the theory I propose does not require that the implicit valuation of the consequences be effect independent. However, if constant bets are constant valuation bets a version of this model, analogous to probabilistic sophistication, is obtained as a special case.

Some modification of the lottery structure is introduced to simplify the presentation. Let $\Theta = \{\theta_1, ..., \theta_n\}$ and, for each $\theta \in \Theta$, let $Z(\theta) = [z_\theta^{**}, z_\theta^{*}]$ be intervals representing *monetary prizes*. Denote by δ_z the (degenerate) lottery that yields the prize z with probability one. Let $\bar{b}^{**} = \left(\delta_{z_{\theta_1}^{**}}, ..., \delta_{z_{\theta_n}^{**}}\right)$ and $\bar{b}^{*} = \left(\delta_{z_{\theta_1}^{*}}, ..., \delta_{z_{\theta_n}^{*}}\right)$ be constant valuation bets that are the maximal and minimal elements of $\mathbb{C}$, respectively.[5]

4.2 Axioms

To obtain the desired generalization, I replace conditional independence, (A.3), that is responsible for the linear structure of the preference relation with two axioms that are analogous to Machina and Schmeidler's [5] axioms 5 and 6.

[5] This assumption is analogous to the assumption of Machina and Schmeidler [5] that the set of outcomes includes a best and worst outcome.

Given $\theta \in \Theta$ and a lottery $p \in \Delta(Z(\theta))$, denote by F_p the cumulative distribution function corresponding to p. Then, as usual, p is said to *first-order stochastically dominate* q if $F_q(z) \geq F_p(z)$ for all $z \in Z(\theta)$.This dominance relation is denoted $p \geq_1 q$. If, in addition, the inequality is strict for some $z \in Z(\theta)$, then the dominance relation is *strict* and is denoted $p >_1 q$.

The axiom of conditional monotonicity asserts that, given $a \in A, b \in B$, and $\theta \in \Theta(a)$, the restriction of $\succcurlyeq$ to $\{(a, b_{-\theta}p) \mid p \in \Delta(Z(\theta))\} \subset \mathbb{C}$ is strictly monotonic with respect to first-order stochastic dominance. Formally,

(A.6) (**Conditional monotonicity)** For all $a \in A$, $\theta \in \Theta$, $b \in B$, and $p, q \in \Delta(Z(\theta))$, if $p \geq_1 q$, then $(a, b_{-\theta}p) \succcurlyeq (a, b_{-\theta}q)$. If $p >_1 q$ and θ is nonnull under a then $(a, b_{-\theta}p) \succ (a, b_{-\theta}q)$.

Let $Y \subseteq T \subseteq \Theta$ then, given $a \in A$ and $\bar{b}'', \bar{b}' \in B^{cv}$ such that $\bar{b}'' \succ \bar{b}'$, a "bet on Y conditional on T" is a bet $\hat{b}$ such that $\hat{b}(\theta) = \bar{b}''(\theta)$ if $\theta \in Y$ and $\hat{b}(\theta) = \bar{b}'(\theta)$ if $\theta \in T$. The next axiom requires that, for every given action, if the decision maker is indifferent between betting on Y conditional on T and betting on the outcome of a coin flip with probability of winning α, also conditional on T, then he must remain indifferent when the constant valuation bets that figure in the bet on Y conditional on T change, and/or when the lotteries on any other effect change. Formally, (A.7) (**Conditional replacement**) For any $a \in A$ and $Y \subseteq T \subseteq \Theta$, if

$$\left(a,\begin{bmatrix}\bar{b}^{**}(\theta) & \theta\in Y\\ \bar{b}^{*}(\theta) & \theta\in T-Y\\ \bar{b}^{*}(\theta) & \theta\in\Theta-T\end{bmatrix}\right)\sim\left(a,\begin{bmatrix}\alpha\bar{b}^{**}(\theta)+(1-\alpha)\,\bar{b}^{*}(\theta) & \theta\in Y\\ \alpha\bar{b}^{**}(\theta)+(1-\alpha)\,\bar{b}^{*}(\theta) & \theta\in T-Y\\ \bar{b}^{*}(\theta) & \theta\in\Theta-T\end{bmatrix}\right)$$

then, for all $\bar{b}', \bar{b}'' \in B^{cv}$ such that $\bar{b}'' \succ \bar{b}'$ and $b \in B$,

$$\left(a,\begin{bmatrix}\bar{b}''(\theta) & \theta\in Y\\ \bar{b}'(\theta) & \theta\in T-Y\\ b(\theta) & \theta\in\Theta-T\end{bmatrix}\right)\sim\left(a,\begin{bmatrix}\alpha\bar{b}''(\theta)+(1-\alpha)\,\bar{b}'(\theta) & \theta\in Y\\ \alpha\bar{b}''(\theta)+(1-\alpha)\,\bar{b}'(\theta) & \theta\in T-Y\\ b(\theta) & \theta\in\Theta-T\end{bmatrix}\right).$$

4.3 Action-dependent subjective probabilities without expected utility

The following theorem asserts that a preference relation satisfies weak order, conditional Archimedean, conditional monotonicity, and conditional replacement if, and only if, there exist a unique family of action-dependent subjective probabilities, $\{\mu(a) \mid a \in A\}$, on Θ and a real-valued utility function, V on $A \times B$ representing $\succcurlyeq$ such that, for any action-bet pair, $(a, b) \in \mathbb{C}$, $(a, b) \to V\left(a, \Sigma_{\theta\in\Theta}\mu(\theta; a)\,\overline{b_{-\theta}b(\theta)}\right)$. The reader will find the following matrix useful in figuring out the meaning of the expression $\Sigma_{\theta\in\Theta}\mu(\theta; a)\,\overline{b_{-\theta}b(\theta)}$. Given a bet b, and $\theta_i \in \Theta$, $\overline{b_{-\theta_i}b(\theta_i)}$, is a constant valuation bet depicted in the corresponding column of the following matrix:

$$\begin{array}{ccccc}
\Theta & \theta_1 & \theta_2 & ... & \theta_n \\
\theta_1 & \bar{b}^1(\theta_1) = b(\theta_1) & \bar{b}^2(\theta_1) & ... & \bar{b}^n(\theta_n) \\
\theta_2 & \bar{b}^1(\theta_2) & \bar{b}^2(\theta_2) = b(\theta_2) & ... & \bar{b}^n(\theta_2) \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
\theta_n & \bar{b}^1(\theta_n) & \bar{b}^2(\theta_n) & ... & b^n(\theta_n) = b(\theta_n)
\end{array}$$

A real-valued function, H, on a convex subset, S, of a linear space is *mixture continuous* if $H(\alpha p + (1-\alpha)q)$ is continuous in α for all $p, q \in S$.

Theorem 3 *Let $\succcurlyeq$ be a preference relation on $\mathbb{C}$. Suppose that $\mathbb{C}$ has a maximal and minimal elements that are constant valuation bets and that assumption (A.0) holds then,*

(a) *The following conditions are equivalent:*

(a.i) *$\succcurlyeq$ satisfies (A.1), (A.2), (A.6), and (A.7).*

(a.ii) *There exist a family of probability measures $\{\mu(a) \mid a \in A\}$ on Θ, and a real-valued function V on $A \times B$, that is mixture continuous in its second argument, such that, for all $(a,b), (a',b') \in \mathbb{C}$,*

$$(a,b) \succcurlyeq (a',b') \Leftrightarrow V\left(a, \Sigma_{\theta\in\Theta}\mu(\theta;a)\,\overline{b_{-\theta}b(\theta)}\right) \geq V\left(a', \Sigma_{\theta\in\Theta}\mu(\theta;a')\,\overline{b_{-\theta}b'(\theta)}\right). \quad (3)$$

Moreover, the family of probability measures $\{\mu(a) \mid a \in A\}$ *on* Θ *is unique and* $\mu(\theta; a) = 0$ *if and only if* θ *is null given* a.

The next Lemma shows that, if the set of constant valuation bets is convex the $V(a, \cdot) = U(\cdot)$ for all $a \in A$.

Lemma 4 *Let* $\succcurlyeq$ *be a preference relation on* $\mathbb{C}$ *satisfying (A.1), (A.2), (A.6), (A.7). Suppose that* $\mathbb{C}$ *has a maximal and minimal elements that are constant valuation bets and that assumption (A.0) holds. If* B^{cv} *is convex then the function* V *in Theorem 3 satisfies* $V(a, \cdot) = U(\cdot)$ *for all* $a \in A$.

4.4 Probabilistically sophisticated choice

In the special case in which constant bets are constant-valuation bets, the model of the preceding section reduces to a probability sophisticated choice model with action-dependent subjective probabilities. Notice that, in this case, $Z(\theta) = Z$ for all $\theta \in \Theta$ and the set of constant bet is convex by definition.

Theorem 5 *Let* $\succcurlyeq$ *be a preference relation on* $\mathbb{C}$. *Suppose that* $\mathbb{C}$ *has a maximal and minimal elements that are constant valuation bets, that assumption (A.0) holds and constant bets are constant valuation bets. Then*

(a) *The following conditions are equivalent:*

(a.i) *$\succcurlyeq$ satisfies (A.1), (A.2), (A.6), and (A.7).*

(a.ii) *There exist a family of probability measures $\{\mu(\cdot; a) \mid a \in A\}$ on Θ, and a mixture continuous, strictly monotonic, real-valued function, V, on $\Delta(Z)$ such that, for all $(a, b), (a', b') \in \mathbb{C}$,*

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow V\left(\sum_{\theta \in \Theta} \mu(\theta; a) b(\theta)\right) \geq V\left(\sum_{\theta \in \Theta} \mu(\theta; a') b'(\theta)\right). \tag{4}$$

Moreover, the family of probability measures $\{\mu(\cdot; a) \mid a \in A\}$ on Θ is unique and $\mu(\theta; a) = 0$ if and only if θ is null given a.

5 Concluding Remarks

Using the device of roulette lotteries, I develop an analytical framework as well as axiomatic subjective expected utility and nonexpected utility models of decision-making under uncertainty that dispense with the notion of states of the nature. In these models the choice set consists of actions-bet pairs, where bets are lottery-valued functions on a set of effects. Decision makers are supposed to believe that they may affect the likelihood of different effects by their choice of actions. The main results of this paper are:

(a) A general subjective expected utility theory with action-dependent subjective probabilities and effect-dependent utilities. The case of effect-independent preferences is obtained as special instances.

(b) A nonexpected utility theory involving well-defined families of action-dependent subjective probabilities on effects and utility representations. The utility assigned to action-bet pairs depends on the convex combination of constant-valuation bets corresponding to the coordinates of the bet, where the weights are the action-dependent subjective probabilities. A probabilistic sophistication version of this model is obtained as a special case in which constant bets are the constant-valuation bets.

6 Proofs

6.1 Proof of Theorem 1

(a) $(a.i) \to (a.ii)$. By the von Neumann–Morgenstern theorem, $\succcurlyeq$ satisfies (A.1), (A.2), and (A.3) if and only if there exists a family of functions $\{w(\cdot,\theta;a) : Z(\theta) \to \mathbb{R} \mid \theta \in \Theta\}$ such that, for every given $a \in A$ and all $b, b' \in B$,

$$(a,b) \succcurlyeq (a,b') \Leftrightarrow \sum_{\theta\in\Theta} \sum_{z\in Z(\theta)} w(z,\theta;a)\, b(z,\theta) \geq \sum_{\theta\in\Theta} \sum_{z\in Z(\theta)} w(z,\theta;a)\, b'(z;\theta). \tag{5}$$

Moreover, $\{\hat{w}(\cdot,\theta;a) : Z(\theta) \to \mathbb{R} \mid \theta \in \Theta\}$ is another set of functions representing $\succcurlyeq$ in the sense of (5), if and only if $\hat{w}(\cdot,\theta;a) = \lambda(a)\, w(\cdot,\theta;a) + \zeta(\theta,a), \lambda(a) > 0$. If θ is null given a then $w(\cdot,\theta;a)$ is a constant function.

Invoking the uniqueness of the functions $w(\cdot,\cdot;a)$, let

$$\sum_{z\in Z(\theta)} w(z,\theta;a)\, \bar{b}^*(z,\theta) = 0 \text{ for all } a \in A \text{ and } \theta \in \Theta. \tag{6}$$

Equation (6) implies that, if θ is null given a then $w(\cdot, \theta; a) = 0$.

By (A.0) $\succcurlyeq$ is nondegenerate. Axiom (A.4) implies that, for all $a, a' \in A$ that are elementarily linked and $\theta \in \Theta(a) \cap \Theta(a')$, there are numbers $\lambda(\theta; a, a') > 0$ and $\zeta(\theta; a, a')$ such that, for all $z \in Z(\theta)$,

$$w(z, \theta; a) = \lambda(\theta; a, a')\, w(z, \theta; a') + \zeta(\theta; a, a'). \tag{7}$$

Equations (6) and (7) imply that, for all $a, a' \in A$ and $\theta \in \Theta(a) \cap \Theta(a')$,

$$\sum_{z \in Z(\theta)} w(z, \theta; a)\, \bar{b}^*(z; \theta) = \sum_{z \in Z(\theta)} [\lambda(\theta; a, a')\, w(z, \theta; a') + \zeta(\theta; a, a')]\, \bar{b}^*(z; \theta) = \zeta(\theta; a, a') = 0. \tag{8}$$

Probabilities: Set $\pi(\theta; a) = 0$ for all $\theta \notin \Theta(a)$. For every $\theta \in \Theta$ let $A(\theta) = \{a \in A \mid \theta \in \Theta(a)\}$. For every $\theta \in \Theta$, fix $a \in A(\theta)$ and let $\{\pi(\cdot; a)\}_{a \in A}$ be a set of probability measures on Θ defined by the solution to the system of equations

$$\pi(\theta; a) - \lambda_{(a,a',\theta)}\pi(\theta; a') = 0, \text{ for all } a' \in A(\theta) - \{a\} \text{ and } \theta \in \Theta(a) \cap \Theta(a') \tag{9}$$

and

$$\sum_{\theta \in \Theta} \pi(\theta; a) = 1 \text{ for all } a \in A. \tag{10}$$

Next I show that the probability measures $\{\pi(\cdot; a)\}_{a \in A}$ on Θ are well-defined. To simplify the exposition let $w_a(b(\theta), \theta) := \sum_{z \in Z(\theta)} w(z, \theta; a)\, b(z, \theta)$ for all $a \in A$, $b \in B$ and $\theta \in \Theta$.

Claim: There exists a unique solution to the system of equations (9) and (10).

Proof. Let $A = \{a_1, ..., a_n\}$ and $\Theta(a) = \{\theta_{1(a)}, ..., \theta_{m(a)}\}$. Write equations (9) and (10) in matrix notation as follows: $\mathbf{M}\boldsymbol{\pi}^t = \boldsymbol{\gamma}$, where

$$\boldsymbol{\pi} = \left(\pi\left(\theta_{1(a_1)}, a_1\right), ..., \pi\left(\theta_{m(a_1)}, a_1\right), ..., \pi\left(\theta_{1(a_n)}, a_n\right), ..., \pi\left(\theta_{m(a_n)}, a_n\right)\right)$$

t denotes the transpose of $\boldsymbol{\pi}$, $\boldsymbol{\gamma}$ is a $\sum_{i=1}^{n} m(a_n)$ column vector whose last n coordinates are 1 and all the other coordinates are 0 and $\mathbf{M}$ is the $\left(\sum_{i=1}^{n} m(a_i)\right) \times \left(\sum_{i=1}^{n} m(a_i)\right)$ matrix of coefficients of corresponding to the system of equations (9) and (10). [6]

Suppose that $\mathbf{M}$ is singular. Take $\bar{b} \in B^{cv}$ and define $\xi_a := \sum_{\theta\in\Theta} w_a\left(\bar{b}(\theta);\theta\right)$, for all $a \in A$. Let

$$\mathbf{w}^t = \left(w_{a_i}\left(\bar{b}\left(\theta_{1(a_i)}\right), \theta_{1(a_i)}\right), ..., w_{a_i}\left(\bar{b}\left(\theta_{m(a_i)}\right), \theta_{m(a_i)}\right)\right)_{i=1}^{n}.$$

Then, by equations (6) and (7), $\mathbf{M}\mathbf{w}^t = \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ a $\sum_{i=1}^{n} m(a_n)$ column vector whose last n coordinates are ξ_{a_i}, $i = 1, ..., n$, and all the other coordinates are 0. Since $\mathbf{M}$ is singular and $\mathbf{w}^t$ exists, there exist $\hat{b} \neq \bar{b}$ such that

$$\hat{\mathbf{w}}^t = \left(w_{a_i}\left(\hat{b}\left(\theta_{1(a_i)}\right), \theta_{1(a_i)}\right), ..., w_{a_i}\left(\hat{b}\left(\theta_{m(a_i)}\right), \theta_{m(a_i)}\right)\right)_{i=1}^{n}$$

satisfies $\mathbf{M}\hat{\mathbf{w}}^t = \boldsymbol{\xi}$ and $\hat{\mathbf{w}}^t \neq \mathbf{w}^t$. Thus

$$\sum_{\theta\in\Theta} w_a\left(\bar{b}(\theta);\theta\right) = \xi_a = \sum_{\theta\in\Theta} w_a\left(\hat{b}(\theta), \theta\right), \ \forall a \in A. \tag{11}$$

[6] For example, if $A = \{a, a'\}$ and $\Theta = \{\theta, \theta'\} = \Theta(a) = \Theta(a')$ then

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & -\lambda(a, a';\theta) & 0 \\ 0 & 1 & 0 & -\lambda(a, a';\theta') \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Equations (5) and (11) imply that $\left(a,\hat{b}\right)\sim\left(a,\bar{b}\right)$ for all $a\in A$. But $\hat{\mathbf{w}}^{t}\neq\mathbf{w}^{t}$ implies that $\hat{b}\left(\theta\right)\notin I\left(\bar{b}\left(\theta\right);\theta,\bar{b},a\right)$ for some $\theta\in\Theta$ and $a\in A$.

Now $\bar{b}\in B^{cv}$ hence, by transitivity, $\hat{b}\in B^{cv}$. This contradicts the uniqueness of constant valuation bets in Definition 1. Hence $\mathbf{M}$ is non-singular and the system of equations (9) and (10) has a unique solution. ■

Utilities: For any given $\theta\in\Theta$, $z\in Z\left(\theta\right)$ and $a\in A$, define $u\left(z;\theta,a\right)=w\left(z,\theta;a\right)/\pi\left(\theta;a\right)$ if $\pi\left(\theta;a\right)>0$ and $u\left(z;\theta,a\right)=\bar{u}$ otherwise. Note that, for all $a\in A$ and $\theta\in\Theta\left(a'\right)\cap\Theta\left(a\right)$,

$$u\left(z;\theta,a'\right)=\frac{w\left(z,\theta;a'\right)}{\pi\left(\theta;a'\right)}=\frac{w\left(z,\theta;a\right)}{\lambda_{(a,a',\theta)}\pi\left(\theta;a'\right)}=\frac{w\left(z,\theta;a\right)}{\pi\left(\theta;a\right)}=u\left(z;\theta,a\right),\tag{12}$$

where the third inequality is implied by (9). Hence $u\left(z;\theta,a\right)=u\left(z;\theta,a'\right):=u\left(z;\theta\right)$ for all $a,a'\in A$ and $\theta\in\Theta\left(a'\right)\cap\Theta\left(a\right)$. By (A.0) any two effects are linked implying that $u\left(z;\theta,a\right)=u\left(z;\theta\right)$ for all $a\in A$ and $\theta\in\Theta\left(a\right)$.

By definition, $w\left(z,\theta;a\right)=\pi\left(\theta;a\right)u\left(z;\theta\right)$ for all $a\in A,\theta\in\Theta$, and $z\in Z\left(\theta\right)$. Define $\succcurlyeq_{a}$ on B as follows, $b\succcurlyeq_{a}b'$ if $\left(a,b\right)\succcurlyeq\left(a,b'\right)$. Then, for each $a\in A$, $\succcurlyeq_{a}$ is represented by the subjective expected utility functional

$$\left(a,b\right)\mapsto\sum_{\theta\in\Theta}\pi\left(\theta;a\right)\sum_{z\in Z(\theta)}u\left(z;\theta\right)b\left(z;\theta\right).\tag{13}$$

Representation: Fix $\bar{a}\in A$ and, for each $a\in A$ define a function $f\left(\cdot,a\right):\mathbb{R}\rightarrow\mathbb{R}$ by

$$\sum_{\theta\in\Theta}\pi\left(\theta;\bar{a}\right)\sum_{z\in Z(\theta)}u\left(z;\theta\right)\bar{b}\left(z;\theta\right)=f\left(\sum_{\theta\in\Theta}\pi\left(\theta;a\right)\sum_{z\in Z(\theta)}u\left(z;\theta\right)\bar{b}\left(z;\theta\right),a\right),\ \forall\bar{b}\in B^{cv}.\tag{14}$$

Then, for all $a \in A$, $f(\cdot, a)$ is well-defined, strictly increasing, continuous function, and $f(\cdot, \bar{a})$ is the identity function.

Next I show that, for all (a, b) and (a', b') in $\mathbb{C}$,

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow f\left(\sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z(\theta)} u(z; \theta) b(z; \theta), a\right) \geq f\left(\sum_{\theta \in \Theta} \pi(\theta; a') \sum_{z \in Z(\theta)} u(z; \theta) b'(z; \theta), a'\right). \tag{15}$$

By (A.0), there is a constant valuation bet $\bar{b}$ such that $(a, b) \succcurlyeq (a, \bar{b}) \sim (a', \bar{b}) \succcurlyeq (a', b')$. By the representation (13),

$$(a, b) \succcurlyeq (a, \bar{b}) \Leftrightarrow \sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z(\theta)} u(z; \theta) b(z; \theta) \geq \sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z(\theta)} u(z; \theta) \bar{b}(z; \theta)$$

and

$$(a', \bar{b}) \succcurlyeq (a', b') \Leftrightarrow \sum_{\theta \in \Theta} \pi(\theta; a') \sum_{z \in Z(\theta)} u(z; \theta) \bar{b}(z; \theta) \geq \sum_{\theta \in \Theta} \pi(\theta; a') \sum_{z \in Z(\theta)} u(z; \theta) b'(z; \theta).$$

But, by equations (14),

$$(a, \bar{b}) \sim (a', \bar{b}) \Leftrightarrow f\left(\sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z(\theta)} u(z; \theta) \bar{b}(z; \theta), a\right) = f\left(\sum_{\theta \in \Theta} \pi(\theta; a) \sum_{z \in Z(\theta)} u(z; \theta) \bar{b}(z; \theta), a'\right).$$

The conclusion follows by transitivity of $\succcurlyeq$ and monotonicity of f_a for all $a \in A$.

This completes the proof that $(a.i) \Rightarrow (a.ii)$.

$(a.ii) \Rightarrow (a.i)$. The fact that $(a.ii)$ implies axioms (A.1), (A.2), and (A.3) follows from the von Neumann–Morgenstern theorem. Axiom (A.4) follows immediately from the fact that $\sum_{z \in Z(\theta)} u(z, \theta) b(z; \theta)$ is independent of a.

(b) Suppose that, for all $\theta \in \Theta$, $v(\cdot,\theta) = \lambda u(\cdot,\theta) + \varsigma(\theta)$, $\lambda > 0$, and $g(\lambda x + \varsigma(a), a) = f(x,a)$, where $\varsigma(a) = \sum_{\theta\in\Theta} \varsigma(\theta)\pi(\theta;a)$. Then, for all $a \in A$,

$$g\left(\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}v(z,\theta)\,b(\theta),a\right) = f\left(\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}u(z,\theta)\,b(\theta),a\right)$$

Hence, (15) implies that $(a,b) \succcurlyeq (a',b')$ if and only if

$$g\left(\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}v(z,\theta)\,b(\theta),a\right) \geq g\left(\sum_{\theta\in\Theta}\pi(\theta;a')\sum_{z\in Z(\theta)}v(z,\theta)\,b'(\theta),a'\right).$$

Let $\{v(\cdot,\theta)\}_{\theta\in\Theta}$ and g be a representation of $\succcurlyeq$ in the sense of $(a.ii)$. Then, by the uniqueness of the additive representation (5) of $\succcurlyeq_a$, for every $a \in A$ and $\theta \in \Theta$, $v(\cdot;\theta)\,\pi(\theta;a) = \lambda(a)\,u(\cdot,\theta)\,\pi(\theta;a) + \zeta(\theta;a)$, $\lambda(a) > 0$. But, by the normalization, $v\left(\bar{b}^*(\theta);\theta\right)\pi(\theta;a) = \zeta(\theta;a)$. Hence $v\left(\bar{b}^*(\theta);\theta\right) = \zeta(\theta;a)/\pi(\theta;a)$. But the left-hand side of the last equation is independent of a, hence $\zeta(\theta;a)/\pi(\theta;a) = \varsigma(\theta)$. Let $\varsigma(a) = \sum_{\theta\in\Theta}\varsigma(\theta)\,\pi(\theta;a)$.

Next observe that, $v(b(\theta),\theta) = \lambda(a)\,u(b(\theta),\theta) + \varsigma(\theta)$. But the left-hand side of the equation is independent of a. Thus $\lambda(a) = \lambda$ for all $a \in A$. Hence, for all $(a,b) \in \mathbb{C}$,

$$g\left(\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}v(z,\theta)\,b(z,\theta),a\right) = g\left(\lambda\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}u(z,\theta)\,b(z,\theta)+\varsigma(a),a\right).$$

But for all $a, a' \in A$ and $\bar{b} \in B^{cv}$,

$$f\left(\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}u(z,\theta)\,\bar{b}(z,\theta),a\right) = f\left(\sum_{\theta\in\Theta}\pi(\theta;a')\sum_{z\in Z(\theta)}u(z,\theta)\,\bar{b}(z,\theta),a'\right)$$

and

$$g\left(\lambda\sum_{\theta\in\Theta}\pi(\theta;a)\sum_{z\in Z(\theta)}u(z,\theta)\,\bar{b}(z,\theta)+\varsigma(a),a\right) = g\left(\lambda\sum_{\theta\in\Theta}\pi(\theta;a')\sum_{z\in Z(\theta)}u(z,\theta)\,\bar{b}(z,\theta)+\varsigma(a'),a'\right).$$

Thus, $g(\lambda x + \varsigma(a), a) = f(x,a)$, for all $a \in A$ and $x \in \{\sum_{\theta\in\Theta} u(\bar{b}(\theta);\theta)\pi(\theta;a) \mid \bar{b} \in B^{cv}\} = \{\sum_{\theta\in\Theta} u(b(\theta);\theta)\pi(\theta;a) \mid b \in B\}$, where the last equality folllows from (A.0).

(c) The uniqueness of $\{\pi(\cdot;a)\}_{a\in A}$ and the fact that $\pi(\theta;a) = 0$ if and only if θ is null given a follows from the definition of the probabilities and the Claim. $\square$

6.2 Proof of Theorem 2

(a) $(a.i) \to (a.ii)$. By Theorem 1, for all $a \in A$, $\theta, \theta' \in \Theta(a)$, and $p, q \in \Delta(Z(\theta)) \cap \Delta(Z(\theta'))$,

$$(a, b_{-\theta}p) \succcurlyeq (a, b_{-\theta}q) \Leftrightarrow \sum_{z\in Z(\theta)} u(z;\theta)\,p(z) \geq \sum_{z\in Z(\theta)} u(z;\theta)\,q(z) \tag{16}$$

and

$$(a, b_{-\theta'}p) \succcurlyeq (a, b_{-\theta'}q) \Leftrightarrow \sum_{z\in Z(\theta')} u(z;\theta')\,p(z) \geq \sum_{z\in Z(\theta')} u(z;\theta')\,q(z). \tag{17}$$

Hence, by axiom (A.5),

$$\sum_{z\in Z(\theta)} u(z;\theta)\,p(z) \geq \sum_{z\in Z(\theta)} u(z;\theta)\,q(z) \Leftrightarrow \sum_{z\in Z(\theta')} u(z;\theta')\,p(z) \geq \sum_{z\in Z(\theta')} u(z;\theta')\,q(z). \tag{18}$$

Thus, by the uniqueness of the von Neumann–Morgenstern utility function, for all $a \in A$ and $\theta \in \Theta(a)$,

$$u(\cdot;\theta) = \sigma(\theta)\,u(\cdot;\theta') + \kappa(\theta),\ \ \sigma(\theta) > 0. \tag{19}$$

Define $u(\cdot;\theta') := u(\cdot)$ then $(a.ii)$ follows from Theorem 1. Hence $(a.i) \to (a.ii)$.

The proof that $(a.ii)$ implies axioms (A.1)–(A.4) follows from Theorem 1. The proof that it also implies axiom (A.5) is straightforward.

The proofs of parts (b) and (c) follow from the corresponding parts of Theorem 1. $\square$

6.3 Proof of Theorem 3

(a) $(a.i) \Rightarrow (a.ii)$. Axiom (A.6) implies that for every given $a \in A$, $\left(a, \bar{b}^{**}\right) \succcurlyeq (a, b) \succcurlyeq \left(a, \bar{b}^{*}\right)$. By (A.0), $\left(a, \bar{b}^{**}\right) \succ \left(a, \bar{b}^{*}\right)$. Hence, by axioms (A.2) and (A.6), for each $(a, b) \in \mathbb{C}$, there exists a unique number $v(a, b) \in [0, 1]$, defined by

$$v(a, b) = Sup\{v \mid (a, b) \succcurlyeq \left(a, \left(v\bar{b}^{**} + (1 - v)\bar{b}^{*}\right)\right)\},$$

such that $(a, b) \sim \left(a, v(a, b)\, \bar{b}^{**} + (1 - v(a, b))\, \bar{b}^{*}\right)$. The proof of the last assertion is by the usual argument (see Kreps [3]). Moreover, axioms (A.2) and (A.6) also imply that, for every $a \in A$, $v(a, \cdot)$ is mixture continuous and monotonic with respect to first order stochastic dominance. Hence, for all $(a, b), (a, b') \in \mathbb{C}$,

$$(a, b) \succcurlyeq (a, b') \Leftrightarrow v(a, b) \geq v(a, b'). \tag{20}$$

By (A.0), for every $(a, b) \in \mathbb{C}$, there is $\bar{b} \in B^{cv}$ such that $(a, b) \sim \left(a, \bar{b}\right)$. Thus $v(a, b) = v\left(a, \bar{b}\right)$. Fix $\bar{a} \in A$ and for all $a \in A$ define a real-valued function $F(\cdot, a)$ on $[0, 1]$ by $F\left(v\left(a, \bar{b}\right), a\right) = v\left(\bar{a}, \bar{b}\right)$, for all $\bar{b} \in B^{cv}$. Then, for all $(a, b), (a', b') \in \mathbb{C}$,

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow F(v(a, b), a) \geq F(v(a', b'), a'). \tag{21}$$

For every $a \in A$ and $\theta \in \Theta$ define $\mu(\theta; a)$ by

$$\left(a, \bar{b}^{*}_{-\theta}\bar{b}^{**}(\theta)\right) \sim (a, \mu(\theta; a)\, \bar{b}^{**} + (1 - \mu(\theta; a))\, \bar{b}^{*}). \tag{22}$$

By the preceding discussion $\mu(\theta; a) \in [0,1]$ is well-defined, and $\mu(\theta; a) = 0$ if and only if θ is null.

By (A.0) $\Theta(a)$ is not empty. Fix $a \in A$ and, without loss of generality, let θ_1 be a nonnull effect under a. Consider two constant valuation bets, $\bar{b}''$ and $\bar{b}'$, such that $\bar{b}'' \succ \bar{b}'$. For every $i = 1, ..., n-1$ and $a \in A$, let $\alpha_i(a)$ be given by

$$\left(a, \bar{b}''(\theta_1), ..., \bar{b}''(\theta_i), \bar{b}'(\theta_{i+1}), ..., \bar{b}'(\theta_n)\right) \sim \left(a, \alpha_i(a)\bar{b}'' + (1-\alpha_i(a))\bar{b}'\right). \tag{23}$$

Then $\alpha_i(a) \in [0,1]$ is well defined, and (by (A.7) with $T = \Theta$) is independent of the constant valuation bets $\bar{b}''$ and $\bar{b}'$ assigned to the subset of effects $\{\theta_1, ..., \theta_i\}$ and $\{\theta_{i+1}, ..., \theta_{i+n}\}$ provided that $\bar{b}'' \succ \bar{b}'$. Let $\alpha_0(a) = 0$, and define

$$\tau_i(a) = \alpha_{n-1}(a)\,\alpha_{n-2}(a)\,...\alpha_i(a)\,(1-\alpha_{i-1}(a)), \quad i = 1, ..., n-1,$$

and

$$\tau_n(a) = (1-\alpha_{n-1}(a)).$$

Then $\Sigma_{i=1}^{n}\tau_i(a) = 1$.

But $\overline{b_{-\theta_i}b(\theta_i)}(\theta_i) = b(\theta_i)$, for all $i = 1, ..., n$. Hence, by repeated application of axiom

(A.7), for any $(a,b)\in\mathbb{C}$,

$$\left(a,\begin{pmatrix} b(\theta_1) & \text{on} & \theta_1 \\ b(\theta_2) & \text{on} & \theta_2 \\ b(\theta_3) & \text{on} & \theta_3 \\ b(\theta_4) & \text{on} & \theta_4 \\ & . & \\ & . & \\ b(\theta_n) & \text{on} & \theta_n \end{pmatrix}\right) \sim \left(a,\begin{pmatrix} \alpha_1(a)\,b(\theta_1)+(1-\alpha_1(a))\overline{b_{-\theta_2}b(\theta_2)}(\theta_1) & \text{on} & \theta_1 \\ \alpha_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}(\theta_2)+(1-\alpha_1(a))b(\theta_2) & \text{on} & \theta_2 \\ b(\theta_3) & \text{on} & \theta_3 \\ b(\theta_4) & \text{on} & \theta_4 \\ . & & . \\ . & & . \\ b(\theta_n) & \text{on} & \theta_n \end{pmatrix}\right) \sim$$

$$\left(a,\begin{pmatrix} \alpha_2(a)\left[\alpha_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}+(1-\alpha_1(a))\overline{b_{-\theta_2}b(\theta_2)}\right](\theta_1)+(1-\alpha_2(a))\,\overline{b_{-\theta_3}b(\theta_3)}(\theta_1) & \text{on} & \theta_1 \\ \alpha_2(a)\left[\alpha_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}+(1-\alpha_1(a))\overline{b_{-\theta_1}b(\theta_2)}\right](\theta_2)+(1-\alpha_2(a))\,\overline{b_{-\theta_3}b(\theta_3)}(\theta_2) & \text{on} & \theta_2 \\ \alpha_2(a)\left[\alpha_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}+(1-\alpha_1(a))\,\overline{b_{-\theta_2}b(\theta_2)}\right](\theta_3)+(1-\alpha_2(a))\,b(\theta_3) & \text{on} & \theta_3 \\ b(\theta_4) & \text{on} & \theta_4 \\ . & & . \\ . & & . \\ b(\theta_n) & \text{on} & \theta_n \end{pmatrix}\right)$$

$$\sim \ldots \sim \left(a,\begin{pmatrix} \tau_1(a)\,b(\theta_1)+\tau_2(a)\,\overline{b_{-\theta_2}b(\theta_2)}(\theta_1)+\cdots+\tau_n(a)\,\overline{b_{-\theta_n}b(\theta_n)}(\theta_1) & \text{on} & \theta_1 \\ \tau_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}(\theta_2)+\tau_2(a)\,b(\theta_2)+\cdots+\tau_n(a)\,\overline{b_{-\theta_n}b(\theta_n)}(\theta_2) & \text{on} & \theta_2 \\ . & & \\ . & & \\ \tau_1(a)\,\overline{b_{-\theta_1}b(\theta_1)}(\theta_n)+\tau_2(a)\,\overline{b_{-\theta_2}b(\theta_2)}(\theta_n)+\cdots+\tau_n(a)\,b(\theta_n) & \text{on} & \theta_n \end{pmatrix}\right) =$$

$$\left(a, \left(\tau_1(a)\overline{b_{-\theta_1}b(\theta_1)} + \tau_2(a)\overline{b_{-\theta_2}b(\theta_2)} + ... + \tau_n(a)\overline{b_{-\theta_n}b(\theta_n)}\right)\right)$$

$$= \left(a, \Sigma_{i=1}^{n}\tau_i(a)\overline{b_{-\theta_i}b(\theta_i)}\right).$$

Thus any action-bet pair (a, b), is indifferent to the action-bet pair $\left(a, \tilde{b}\right)$ where $\tilde{b}$ is a convex combination of the constant valuation bets corresponding to the coordinates of the original bet b. Hence

$$\left(a, \bar{b}^{**}_{-\theta_j}\bar{b}^{*}\right) \sim (a, \tau(\theta_j; a)\bar{b}^{**} + (1 - \tau(\theta_j; a))\bar{b}^{*}), \tag{24}$$

where $\tau(\theta_j; a) = \Sigma_{i \neq j}\tau_i(a)$. But this implies that, $\tau(\theta_j; a) = \mu(\theta_j; a)$ for all θ_j, $j = 1, ..., n$, and $a \in A$, and $(a, b) \sim \left(a, \Sigma_{\theta \in \Theta}\mu(\theta; a)\overline{b_{-\theta}b(\theta)}\right)$.

Define $V\left(a, \Sigma_{\theta \in \Theta}\mu(\theta; a)\overline{b_{-\theta}b(\theta)}\right) = F(v(a, b), a)$. Then, by (21), for all $(a, b), (a', b') \in \mathbb{C}$,

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow V\left(a, \Sigma_{\theta \in \Theta}\mu(\theta; a)\overline{b_{-\theta}b(\theta)}\right) \geq V\left(a', \Sigma_{\theta \in \Theta}\mu(\theta; a')\overline{b'_{-\theta}b'(\theta)}\right). \tag{25}$$

This completes the proof that $(a.i) \Rightarrow (a.ii)$. The proof that $(a.ii) \Rightarrow (a.i)$ is immediate.

To prove the uniqueness of $\{\mu(\cdot; a) \mid a \in A\}$, let $\{\eta(\cdot; a) \mid a \in A\}$ be another family of probability measures on Θ and W a real-valued function on $A \times B$ representing $\succcurlyeq$ in the sense of (25). Then, for some $a \in A$ and $\theta \in \Theta$, $\eta(\theta; a) > \gamma > \mu(\theta; a)$. Consider the action-bet pairs $\left(a, \bar{b}^{**}_{-\theta}\bar{b}^{*}\right)$ and $\left(a, \gamma\bar{b}^{**} + (1 - \gamma)\bar{b}^{*}\right)$. Note that

$$\eta(\theta; a)\bar{b}^{**} + (1 - \eta(\theta; a))\bar{b}^{*} >_1 \gamma\bar{b}^{**} + (1 - \gamma)\bar{b}^{*} >_1 \mu(\theta; a)\bar{b}^{**} + (1 - \mu(\theta; a))\bar{b}^{*}.$$

Since $W(a, \cdot)$ is strictly monotonic increasing with respect to first order stochastic dominance, we have

$$W\left(a,\eta\left(\theta;a\right)\bar{b}^{**}+\left(1-\eta\left(\theta;a\right)\right)\bar{b}^{*}\right)>W\left(a,\gamma\bar{b}^{**}+\left(1-\gamma\right)\bar{b}^{*}\right). \tag{26}$$

This implies $\left(a,b_{-\theta}^{**}b^{*}\right)\succ\left(a,\gamma b^{**}+\left(1-\gamma\right)\bar{b}^{*}\right)$. By the same argument,

$$V\left(a,\mu\left(\theta;a\right)\bar{b}^{**}+\left(1-\mu\left(\theta;a\right)\right)\bar{b}^{*}\right)<V\left(a,\gamma\bar{b}^{**}+\left(1-\gamma\right)\bar{b}^{*}\right). \tag{27}$$

Thus $\left(a,\gamma b^{**}+\left(1-\gamma\right)\bar{b}^{*}\right)\succ\left(a,\bar{b}_{-\theta}^{**}\bar{b}^{*}\right)$. A contradiction. □

6.4 Proof of Lemma 4

Suppose that $\succcurlyeq$ satisfies (A.1), (A.2), (A.6), (A.7) and B^{cv} is a convex set. By Theorem 3, $\succcurlyeq$ is represented by $(a,b)\mapsto V\left(a,\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)$. But B^{cv} is convex, hence $\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\in B^{cv}$. Thus, by definition, for all $a,a'\in A$, $\left(a,\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)\sim\left(a',\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)$. Hence, by Theorem 3, $V\left(a,\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)=V\left(a',\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)$ $U\left(\Sigma_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)$. □

6.5 Proof of Theorem 5

$(a)\,(a.i)\Rightarrow(a.ii)$. Since constant bets are constant valuation bets, $\overline{b_{-\theta}p}=[p,...,p]$ for all $\theta\in\Theta$ and, by definition, B^{cv} is a convex set. Thus, for the purpose of this proof, the constant valuation bet $\overline{b_{-\theta}p}$ is identified with p.

By Lemma 4 the preference relation $\succcurlyeq$ on $\mathbb{C}$ is represented by $V\left(\sum_{\theta\in\Theta}\mu\left(\theta;a\right)\overline{b_{-\theta}b\left(\theta\right)}\right)$, where for every $a\in A$, $\mu\left(\cdot;a\right)$ is a unique probability measure on Θ. But $\overline{b_{-\theta}b\left(\theta\right)}=$

$[b(\theta), ..., b(\theta)]$. Hence, by the identification above, $\sum_{\theta\in\Theta} \mu(\theta; a) \overline{b_{-\theta} b(\theta)} = \sum_{\theta\in\Theta} \mu(\theta; a) b(\theta)$.

Thus, for all $(a, b), (a', b') \in \mathbb{C}$

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow V\left(\sum_{\theta\in\Theta} \mu(\theta; a) b(\theta)\right) \geq V\left(\sum_{\theta\in\Theta} \mu(\theta; a') b'(\theta)\right). \tag{28}$$

The other parts of the theorem follow from the proof of Theorem 3. □

References

[1] Anscombe, F., Aumann, R. A definition of subjective probability, Annals of Mathematical Statistics **34**, 199-205 (1963).

[2] Karni, E. Subjective expected utility theory without states of the world, Journal of Mathematical Economics, forthcoming (2006) .

[3] Kreps, D. Notes on the Theory of Choice. Boulder, Westview Press 1988.

[4] Machina, M. J., Schmeidler, D. A more robust definition of subjective probability, Econometrica **60**, 745–780 (1992).

[5] Machina, M. J., Schmeidler, D. Bayes without Bernoulli: Simple conditions for probabilistically sophisticated choice, Journal of Economic Theory **67**, 106-128 (1995).

[6] Savage, L. J. The foundations of statistics. New York: John Wiley and Sons, 1954.