A Note on Instability and Indeterminacy in Search and Matching Models

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Abstract

We demonstrate the possibility of indeterminacy and non-existence of equilibrium dynamics in a standard business cycle model with search and matching frictions in the labor market. Our results arise for empirically plausible parametrizations and do not rely upon a mechanism such as increasing returns.

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KEYWORDS: Search and matching, indeterminacy

1 Introduction

The search and matching model of Mortensen and Pissarides (1994) has become a popular and successful framework for analyzing labor market dynamics in dynamic stochastic general equilibrium (DSGE) models.[1] In this paper, we point out two potential pitfalls that can arise in the specification and solution of this class of models. Depending on the stylized facts chosen for the calibration procedure, the model may imply values that are internally inconsistent and implicitly violate other stylized facts. Secondly, we show that the model solution can be non-existent or indeterminate for plausible parameter values. In particular, uniqueness problems arise when endogenous matching in response to labor market pressures is too elastic or not elastic enough. In such a scenario, extraneous uncertainty, 'sunspots', can lead to business cycle fluctuations.

Indeterminacy in search and matching models has previously been addressed by Giammarioli (2003). This paper differs from ours in that it introduces increasing returns in the matching function. We show that indeterminacy arises even under constant returns.

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[1] A non-exhaustive list of references includes Merz (1995), Andolfatto (1996), Cooley and Quadrini (1999), Chéron and Langot (2000), den Haan et al. (2000), Krause and Lubik (2003), Walsh (2003), Trigari (2004).

Our paper is closer to Burda and Weder (2002) in that respect. Their indeterminacy results are driven, however, by the existence of labor market distortions, such as taxes, and associated policy functions.

This note proceeds as follows. We present a canonical DSGE model with search and matching frictions in the next section. Section 3 discusses issues related to the calibration of this model, while section 4 derives its determinacy properties. The final section briefly summarizes and concludes.

2 A Canonical DSGE Model of Labor Market Search and Matching

We present a bare-bones version of a discrete-time DSGE framework with search and matching frictions. New employment relationships are the result of time-consuming search. Existing matches are subject to job destruction, which leads to a flow of workers into the unemployment pool. The behavior of the aggregate economy is governed by the choices of a representative household. These assumptions lead to steady-state and dynamic equations that are amenable to analytical solutions. The presentation of the model is standard. We refer to Krause and Lubik (2003) for additional discussion and references.

There is a continuum of identical firms that employ workers each of whom inelastically supplies one unit of labor.[2] Time is discrete. Output y of a typical firm is linear in employment n:

$$y_t = n_t. \tag{1}$$

The matching process is represented by a constant returns matching function, $m(u_t, v_t) = m u_t^{\xi} v_t^{1-\xi}$, of unemployment u and vacancies v, with parameters $m > 0$ and $0 < \xi < 1$. Unemployment is defined as:

$$u_t = 1 - n_t. \tag{2}$$

Inflows to unemployment arise from exogenous job destruction at rate $0 < \rho < 1$. Employment therefore evolves according to:

$$n_{t+1} = (1-\rho)[n_t + m(u_t, v_t)]. \tag{3}$$

We can define $q(\theta_t)$ as the probability of filling a vacancy, or the firm-matching rate. $\theta_t = v_t/u_t$ is labor market tightness. In terms of the matching function, we can write this as $q(\theta_t) = m(u_t, v_t)/v_t = m\theta_t^{-\xi}$. Similarly, the probability of finding a job, the worker-matching rate, is $p(\theta_t) = m(u_t, v_t)/u_t = m\theta_t^{1-\xi}$.

Firms maximize profits, using a discount factor $\beta^t \frac{\lambda_t}{\lambda_0}$ (to be determined below):

$$\max_{\{v_t, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \left[n_t - w_t n_t - c v_t\right] + \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0} \mu_t \left[(1-\rho)[n_t + v_t q(\theta_t)] - n_{t+1}\right]. \tag{4}$$

Wages paid to the workers are w, while $c > 0$ is a firm's cost of opening a vacancy. Firms decide on how many vacancies to post (which can be turned into employment relationships)

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[2] For expositional convenience, we present the problem of a representative firm.

and how many workers to hire. The first order conditions are:

$$n_t \quad : \qquad \mu_t = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[1 - w_{t+1} + \mu_{t+1}(1-\rho) \right], \tag{5}$$

$$v_t \quad : \qquad c = \mu_t (1-\rho) q(\theta_t), \tag{6}$$

which imply a *job creation* condition:

$$\frac{c}{q(\theta_t)} = (1-\rho)\beta \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left[1 - w_{t+1} + \frac{c}{q(\theta_{t+1})} \right]. \tag{7}$$

We assume that the economy is populated by a representative household. The household is endowed with one unit of labor which is supplied inelastically to the labor market. The optimization problem of the household is:

$$\max_{\{C_t\}_{t=0}^{\infty}} U = \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} + \chi_t b - (1-\chi_t) \right], \tag{8}$$

subject to:

$$C_t = Y_t, \tag{9}$$

where C is consumption and Y is income earned from labor and and residual profits from the firms; $0 < \beta < 1$ is the discount factor, and σ^{-1} is the intertemporal elasticity of substitution. If unemployed, the indicator χ_t equals one, and the household enjoys utility b. While employed $\chi_t = 0$, and the household suffers disutility normalized to one.[3] Perfect risk-sharing within the households implies that firms use the households' intertemporal rate of substitution to evaluate their profit streams. From the household's (trivial) first order condition we find that $\lambda_t = C_t^{-\sigma}$. The income that accrues to households is:

$$Y_t = y_t - c v_t, \tag{10}$$

as resources are lost in the search process.

If wages are set according to the Nash bargaining solution, it is straightforward to show that they are given by:

$$w_t = \eta \left(1 + c\theta_t \right) + (1-\eta) b. \tag{11}$$

With this wage equation, and inserting consumption for the Lagrange multipliers, the job creation condition becomes:

$$\frac{c}{q(\theta_t)} = (1-\rho)\beta \frac{Y_t^{\sigma}}{Y_{t+1}^{\sigma}} \left[(1-\eta) - \eta c \theta_{t+1} - (1-\eta) b + \frac{c}{q(\theta_{t+1})} \right]. \tag{12}$$

The dynamics of the model are given by the five equations in five unknowns: (2), (3), (10), (12), and the definition of labor market tightness θ_t.

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[3] We assume income pooling between employed and unemployed households, and abstract from potential incentive problems concerning labor market search. This allows us to treat the labor market separate from the consumption choice. See Merz (1995) and Andolfatto (1996) for discussion of these issues.

3 Steady State and Calibration

Analysis of DSGE models proceeds by first computing the non-stochastic steady state and then linearizing the dynamic equation system around it. In order to make quantitative statements, parameter values have to be determined either directly from a priori information or indirectly from calibration. The equations describing the steady state are:

$$u = 1 - n, \tag{13}$$

$$\theta = \frac{v}{u}, \tag{14}$$

$$n = \frac{1-\rho}{\rho} m v^{1-\xi} u^{\xi}, \tag{15}$$

$$Y = n - cv, \tag{16}$$

$$(1-\eta)(1-b) = \frac{1-\beta(1-\rho)}{\beta(1-\rho)} \frac{c}{m} \theta^{\xi} + c\eta\theta. \tag{17}$$

There are five endogenous variables (u, n, v, θ, y) and seven structural parameters (ρ, m, ξ, c, β, ,η, b). Note that because of the non-linearity in the last equation there is no analytical solution to this system. Given values for the parameters, however, we can compute a numerical solution. Using a non-linear equation solver we determine θ from Eq. (17), which depends on all parameters.[4] From Eq. (15) we can find $u = \left(1 + \frac{1-\rho}{\rho} m \theta^{1-\xi}\right)^{-1}$, and the other variables follow immediately.

In a sense, this is the preferred procedure since it maps structural parameters into (potentially) observable variables whose behavior which can be used to evaluate the model's performance. The problem is that more often than not pertinent information may not be available for some or most parameters. In this case, we may be able to compute these indirectly from the steady-state values of quantifiable endogenous variables, in other words, by calibration. For instance, the weight on capital in an aggregate Cobb-Douglas production function can be inferred from the output share accruing to capital services. Parameters governing disutility form working can be calibrated from empirical studies on labor supply elasticities, and so on. This strategy is even more warranted in models where there is virtually no independent information about specific parameters. In the search and matching model, this would include the match parameter m, the Nash parameter η and the disutility parameter b.

We could calibrate the model by setting the unemployment rate $u = \overline{u}$ to be consistent with the average found in the data. This implies that one parameter has to be solved for endogenously. Additionally, from evidence on the rates at which firms fill vacancies, we can fix the matching rate $\overline{q} = m\theta^{-\xi}$. Hence, another parameter has to be determined endogenously. Using $n = 1 - \overline{u}$ in (15) we find that $m = \left(\frac{\rho}{1-\rho}\frac{1-\overline{u}}{\overline{u}}\right)^{\xi} \overline{q}^{1-\xi}$ and $\theta = \left(\frac{m}{\overline{q}}\right)^{1/\xi}$. From (17) we can then also compute: $\frac{1-b}{c} = \frac{\eta}{1-\eta}\theta + \frac{1}{1-\eta}\frac{1-\beta(1-\rho)}{\beta(1-\rho)}\frac{\theta^{\xi}}{m}$. Note, however, that this condition does not pin down b and c independently, nor does any other restriction in the model. Eq. (16) helps only insofar it restricts c such that y remains positive.[5]

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[4] Since the function in θ is monotonically increasing for non-negative θ there is a unique solution to this equation as long as $0 \leq b < 1$.

[5] To be precise, this is not a shortcoming of the calibration strategy per se. The model does not offer any

An alternative is to use evidence on worker matching rates: $\overline{p} = m\theta^{1-\xi}$. We can again use (15), which implies the condition: $\overline{p} = \frac{\rho}{1-\rho}\frac{1-\overline{u}}{\overline{u}}$. This is a contradiction since ρ is a structural parameter and the unemployment rate $\overline{u}$ has been calibrated. In other words, given $\overline{u}$ and ρ, calibrating the worker matching rate violates the model's cross-equation restrictions! To give a numerical example, we set the unemployment rate to 6%, $\overline{u} = 0.06$, and $\rho = 0.1$, so that $p = 1.74$. In contrast, a commonly used value (e.g. by den Haan et al., 2000) for the matching rate $\overline{p}$ is 0.6. A related mistake would be to compute θ directly from the matching rates. Setting $\overline{q} = 0.7$ and $\overline{p} = \theta\overline{q} = 0.6$ leads to $\theta = 0.857$. This is inconsistent with the value for θ when we start out with the firm matching rate, which yields $\theta = 2.487$. Again, the calibration disregards the model's structure and imposes additional and arbitrary restrictions. Joint information on matching rates may be useful in other setups, but in the standard model they are not mutually independent. Evidence on the worker matching rate can be used in the calibration, but this implies that either ρ or u are no longer exogenous. With $\overline{u} = 0.06$ ($\rho = 0.1$) a matching rate of $\overline{p} = 0.6$ requires that $\rho = 0.037$ ($\overline{u} = 0.156$). Intuitively, the more jobs are destroyed, the higher has to be steady state unemployment to preserve aggregate transition rates into employment. Finally, θ can then be found from (15) by calibrating m.

4 Indeterminacy and Non-Existence

We now proceed by linearizing the relevant equations around the steady state. It is well known that dynamic models can have multiple equilibria, or that the solution may not even exist. We show that both scenarios are quite possible in the standard search and matching model. The linearized system is as follows (with $\widehat{x}_t = \log x_t - \log x$):

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\begin{aligned}
u\,\widehat{u}_t &= -n\,\widehat{n}_t, && (18)\\
\widehat{\theta}_t &= \widehat{v}_t - \widehat{u}_t, && (19)\\
\widehat{n}_{t+1} &= (1-\rho)\widehat{n}_t + \rho(1-\xi)\widehat{v}_t + \rho\xi\widehat{u}_t, && (20)\\
\widehat{Y}_t &= \frac{n}{y}\widehat{n}_t - \frac{cv}{y}\widehat{v}_t, && (21)\\
\xi\widehat{\theta}_t - \sigma\widehat{Y}_t &= \left(\frac{c\xi}{m}\theta^{\xi} - \eta c\theta\right)X^{-1}\widehat{\theta}_{t+1} - \sigma\widehat{Y}_{t+1}, && (22)
\end{aligned}
$$

where $X = \frac{1}{\beta(1-\rho)}\frac{c}{m}\theta^{\xi}$. It is straightforward to substitute out $\widehat{u}_t$, $\widehat{v}_t$, and $\widehat{Y}_t$, so that we are left with:

$$
\begin{bmatrix} \widehat{\theta}_{t+1} \\ \widehat{n}_{t+1} \end{bmatrix} = \begin{bmatrix} \frac{\xi+\sigma\frac{cv}{y}}{\alpha_1} + \rho(1-\xi)\frac{\alpha_2}{\alpha_1} & -\frac{\alpha_2}{\alpha_1}\frac{\rho}{u} \\ \rho(1-\xi) & \frac{u-\rho}{u} \end{bmatrix} \begin{bmatrix} \widehat{\theta}_t \\ \widehat{n}_t \end{bmatrix}, \qquad (23)
$$

where $\alpha_1 = \beta(1-\rho)(\xi - \eta m\theta^{1-\xi}) + \sigma\frac{cv}{y}$ and $\alpha_2 = \sigma\frac{n}{y}(1+c\theta)$. This reduced form is expressed in terms of the state (or predetermined) variable $\widehat{n}_t$ and the jump variable $\widehat{\theta}_t$. The stability properties of the solution depend on the eigenvalues of the coefficient matrix. A unique solution requires that one root be inside the unit circle and the other root outside.

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cross-equation restrictions that would identify either of these parameters. In a model with varying labor supply this would be different. The point is, however, that this situation can arise in any DSGE model since most likely some economic action will always be exogenously determined.

Indeterminacy arises when both roots are inside the unit circle, while non-existence occurs with both roots being explosive. The coefficient matrix is sufficiently complicated to prevent derivation of analytical results. For illustrative purposes, we therefore make the simplifying assumption that the representative household is risk-neutral, $\sigma = 0$ (simulation results are presented for the general case). Under this assumption, the coefficient matrix reduces to:

$$\begin{bmatrix} \frac{\xi}{\beta(1-\rho)(\xi-\eta p)} & 0 \\ \rho(1-\xi) & \frac{u-\rho}{u} \end{bmatrix}. \tag{24}$$

Since the matrix is triangular the eigenvalues can be read off the principal diagonal. Note further that the worker matching rate $p = m\theta^{1-\xi}$. As shown above, this is also equal to $p = \frac{\rho}{1-\rho}\frac{1-u}{u}$. Depending on the calibration strategy the coefficient matrix therefore may contain both exogenous and endogenous components which we have to take into account when deriving analytical results.

We establish the determinacy properties in the following proposition.

Proposition 1 *1. The model solution is indeterminate if and only if*

   (a) $0 < \rho < 2u$,
   (b) $0 < \xi < \frac{\beta(1-\rho)}{1+\beta(1-\rho)}\eta p$.

2. The model solution is non-existent if and only if

   (a) $\rho > 2u > 0$,
   (b) $\frac{\beta(1-\rho)}{1+\beta(1-\rho)}\eta p < \xi < 1$.

3. The model solution is unique if and only if either

   (a) $0 < \rho < 2u$,
   (b) $\frac{\beta(1-\rho)}{1+\beta(1-\rho)}\eta p < \xi < 1$,

   or

   (c) $\rho > 2u > 0$,
   (d) $0 < \xi < \frac{\beta(1-\rho)}{1+\beta(1-\rho)}\eta p$.

Proof. Indeterminacy requires both roots inside the unit circle. Call $\lambda_2 = \frac{u-\rho}{u}$. It is straightforward to verify that $|\lambda_2| < 1$ over the permissible range iff $0 < \rho < 2u$. Call the other root $\lambda_1 = \frac{\xi}{\beta(1-\rho)(\xi-\eta p)}$. We have to distinguish two cases: if $\xi > \eta p$ no parameter combination can be found such that $|\lambda_1| < 1$. If $\xi < \eta p$ we can write $-\beta(1-\rho)(\xi-\eta p) > \xi > \beta(1-\rho)(\xi-\eta p)$. Simple algebra in combination with $\xi > 0$ then yields $1(b)$. Non-existence requires that both roots be outside the unit circle. This is just the opposite scenario discussed before. Part 2 of the proposition follows immediately. Uniqueness requires one stable and one unstable eigenvalue. The parameter regions are consequently implied by those not considered in Part 1 and 2. ■

The Proposition shows that indeterminacy is a plausible outcome in this model. It arises when the job destruction rate is less than twice the (calibrated) unemployment rate. A commonly used value for ρ is 10%, so that the unemployment rate has to be less than 5%. It could be argued that u not only measures the registered unemployed but all workers potentially available for employment, such as discouraged workers. u should therefore be assigned a much higher value ruling out non-existence of the equilibrium.

Condition $1(b)$ imposes an upper bound on the match elasticity ξ. Under a standard calibration with $\beta = 0.98$, $\rho = 0.1$, $u = 0.15$, and $\eta = 0.5$, this upper bound is 0.147. Since ξ is typically calibrated to be around 0.5 indeterminacy appears not to be a likely outcome. Note, however, that Cooley and Quadrini (1999) argue that a low elasticity in the range of $\xi = 0.1$ is necessary to match labor market cyclicality. This can be justified by an increasing availability of out-of-the labor force workers when aggregate conditions improve. In the case with constant job destruction this calibration would lead to indeterminacy! Furthermore, the upper bound is increasing in the Nash-bargaining parameter. But even if $\eta \rightarrow 1$, indeterminacy would not occur for standard parameter choices.

Suppose now that the unemployment rate is set to $u = 0.06$. In this case, the upper bound increases to 0.816, which would imply indeterminacy for typical search elasticity choices. Similarly, the use of both worker and firm matching rates leads to an inconsistency in terms of θ. It may very well be that this incorrect calibration (which imposes, for instance, $p = 0.6$) implies a unique equilibrium, whereas the correct calibration (where $p = 1.74$ would have substantially increased the upper bound in $1.(b)$) results in indeterminacy.

Results for the case with a risk-averse household ($\sigma > 0$) are presented in Figure 1. We have plotted determinacy regions for different subsets of the parameter space. The simulation is based on the benchmark calibration where $\beta = 0.98$, $c = 0.1$, $\sigma = 1$, $\eta = 0.5$, $\xi = 0.5$, $q = 0.7$, $u = 0.12$, $\rho = 0.1$. We depict results for combinations of ξ and other parameters. As a general conclusion, determinacy problems tend to arise when the match elasticity ξ is either too small or too big. For small ξ the equilibrium is indeterminate when the job destruction rate ρ or the unemployment rate u are too small. This is related to the analytical condition found in Proposition 1. Furthermore, firm-matching rates q above 0.2 and a Nash-parameter that puts more weight on workers also lead to multiplicity.

No equilibrium exists for large ξ and either a small unemployment rate or ρ above 0.2. We also analyzed the sensitivity of the regions with respect to σ. As $\sigma \rightarrow 0$ the indeterminacy regions expand. In particular any q implies multiple equilibria when $\xi < 0.2$. In the limit the boundaries between regions are given in the Proposition. As the household becomes more risk averse regions of indeterminacy disappear entirely.

5 Conclusion

We have demonstrated two (related) issues in the analysis of search and matching models within a DSGE framework. First, calibration of structural parameters can impose artificial restrictions on the steady state that violate the model's implied cross-equation restrictions. While this is an obvious point that careful analysis can avoid, in practice these limitations can be obscured by the highly complex and non-linear nature of this class of models. Simply taking parameter values from previous studies may lead to inconsistencies. Secondly, we show that for reasonable parameterizations the model has multiple equilibria so that extraneous uncertainty, i.e. animal spirits, can cause business cycles. We have identified the match elasticity as a crucial parameter. Since it governs the volatility of job creation, matching the data may require values that imply indeterminacy.

These properties are obviously model specific, but our conclusions are likely robust to modifications such as endogenous job destruction. While the boundaries of the determinacy regions are likely to shift, the dynamic mechanism stays unaffected. Naturally, this is a topic for further investigation. Moreover, researchers may actually be interested in the business cycle implications of indeterminacy that does not depend on policy rules or externalities. It appears plausible that actual labor market decisions are characterized to some extent by animal spirits. Further research should shed some light on this issue.

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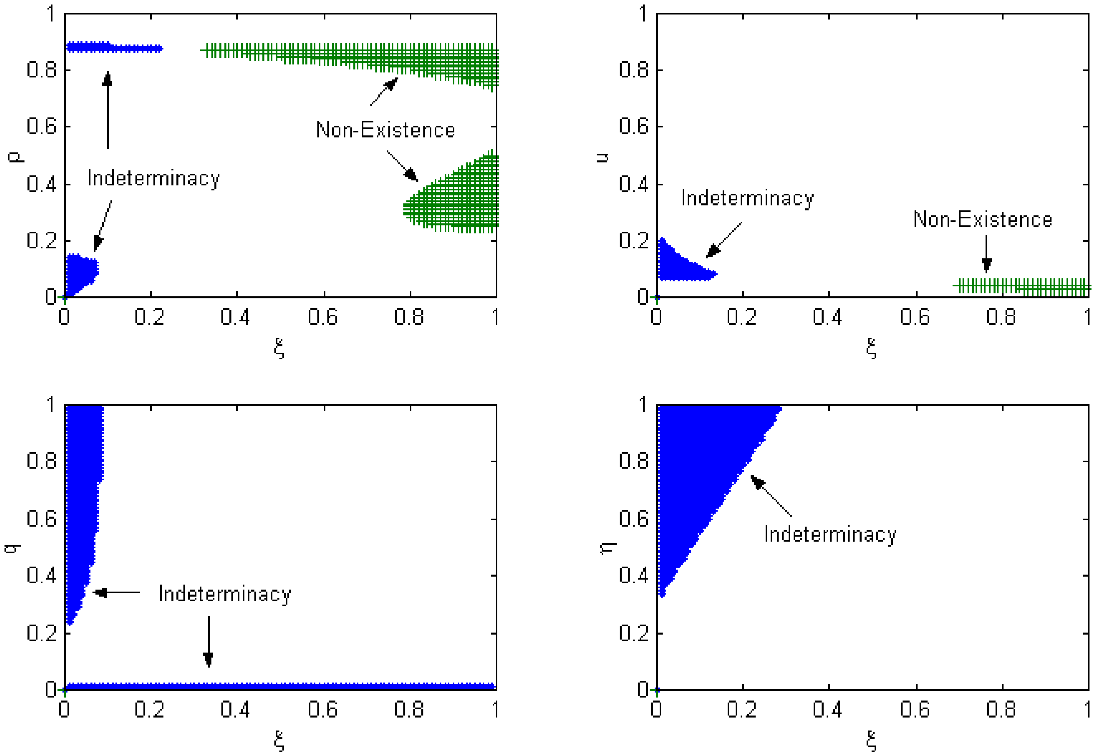

Figure 1: Determinacy Regions