Some Foundations for Multiplicative Habits Models

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Abstract

While consumption models with multiplicative habits are becoming increasingly popular, some important theoretical questions about these models have not yet been addressed. This paper fills three such gaps: Existence of an optimal consumption path; satisfaction, by that path, of the consumption Euler equation; and convergence of that path to the stationary (steady state) path.

1 Introduction

Economic models with habit formation are now becoming popular. However, mathematical foundations of such models are incomplete. The existing literature[2] has proceeded by making several implicit assumptions about the solution to habits models which have not been proven. This paper examines the circumstances under which these assumptions are justified, and finds parametric restrictions that must be imposed for some of the conditions to hold true.

The first problem is that we cannot assume the existence of the optimal

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[2]A related paper by Alonso-Carrera et al. (2003) claims that the optimal interior path exists under several conditions in a discrete time infinite horizon model. They show that the interior path is optimal if the solution of the difference equations which is derived from the first order conditions is unique, strictly positive, converges to a strictly positive stationary path and satisfies the transversality condition. The problem is that they have not shown that the conditions they impose hold true for the problem in question and in fact one of the conditions they assume (uniqueness) is not true.

path without proof as Carroll et al. (2000) or Francisco et al. (2004) did, for example, because the utility function with habits is nonbounded, nonconcave and time nonseparable. Furthermore, even if such an optimal path exists, the path may not be the interior path satisfying the Euler equation.

The second problem is that there may be multiple Euler paths and then the interior path that converges to the stationary growth path might not be optimal. In fact, Carroll (2000) and Francisco et al. (2004) find two paths which satisfy the Euler equation. Furthermore, as far as I know, nothing in the existing literature rules out oscillatory paths. Indeed, Benhabib and Nishimura (1985) show that two period cycles can be optimal in a dynamic model which is similar to our model.

The third problem is that we cannot assume without proof that the optimal path must satisfy the transversality condition. Carroll et al. (2000) and Francisco et al. (2004) claim that one of the interior paths satisfying the Euler equation is not optimal because the path violates the transversality condition. Since the utility function they are using is nonconcave and time nonseparable, the optimal path may not satisfy the condition.

Here we first show that the optimal path does exist[3] and it satisfies the Euler equation in a discrete time infinite horizon economic growth model. The model can easily be applied to the consumption model with multiplicative habits in Carroll (2000). Next we show that in AK growth model, the optimal path converges to stationary growth path under one restriction on parameters. Furthermore, the convergence of habit stocks is monotone. Finally we prove that the transversality condition with respect to capital stock is necessary condition and derive necessary terminal conditions on habit stock.

[3] Zapatero and Palmero (2003) study the Bellman equation with nonconcave and nonbounded utility function, and show that under several assumptions we can use "maximum" in the Bellman operator instead of "supremum". But it is not clear that our model satisfies all these assumptions.

2 Existence of the optimal interior path

2.1 Model

Here we will study the neoclassical growth model with a concave production function. The problem can be written as:

$$\sup\left\{\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t},h_{t}\right)\right\} \tag{1}$$

subject to:

$$\begin{aligned}
k_{t+1} &= y_t - c_t + (1-\delta)\,k_t && (2)\\
y_t &= Ak_t^{\alpha} && (3)\\
h_{t+1} &= (1-\lambda)\,h_t + \lambda c_t && (4)\\
(k_0,h_0) &= (k,h)\in\mathbb{R}^2_{++}\ \text{, given} && (5)\\
k_t &\geq 0 && (6)
\end{aligned}$$

where y_t is the income, δ is the depreciation rate, c_t is the consumption, k_t is the capital stock, h_t is the consumption habit stock and β is the discount rate. k_t satisfies:

$$k_{t+1} = F\left(k_t\right) - c_t \tag{7}$$

where $F\left(k\right) = Ak^{\alpha} + (1-\delta)\,k$. Parameters satisfy $A > 0$, $\alpha \in (0,1]$, $\delta \in [0,1)$, $\beta \in (0,1)$ and $\lambda \in (0,1]$. We also assume that if $\alpha = 1$, the depreciation rate satisfies $A + (1-\delta) > 1$.

The utility function is given by:

$$\begin{aligned}
u\left(c,h\right) &= \frac{1}{1-\rho}\left(\frac{c}{h^{\gamma}}\right)^{1-\rho} && (8)\\
&= \frac{1}{1-\rho}\left\{c^{1-\gamma}\left(\frac{c}{h}\right)^{\gamma}\right\}^{1-\rho}
\end{aligned}$$

with $\rho > 1$ and $0 < \gamma < 1$. Clearly u is not concave[4].

2.2 Properties of the value function

Let the value function V be $V(k,h) = \sup\left\{\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)\right\}$. For showing that the consumption path which attains the supremum exists and the path is interior, we prove several properties of the value function. Below we call an allocation $\{c_t, k_t, h_t\}_{t=0}^{\infty}$ with initial stock $(k_0, h_0) = (k, h)$ which satisfies $(2), (4)$, (5) and (6) as a feasible allocation.

First, let us show that the value function is negative. Since $u(c_t, h_t) < 0$, $\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) \le u(c_0, h)$ for all feasible allocations. Furthermore, $c_0 \le F(k)$ for every allocation. It follows that:

$$V(k,h) \le u(F(k), h)$$

Since $(F(k), h) \in \mathbb{R}_{++}^2$, $u(F(k), h) < 0$ and then $V(k,h) < 0$. Furthermore, since $\lim_{k\to 0} u(F(k), h) = -\infty$, we obtain:

$$\lim_{k\to +0} V(k,h) = -\infty$$

Second, we will show that $V(k,h) > -\infty$. Let $k^* > 0$ be such that $k^* \le k$ and $k^* < F(k^*)$. We can easily check that such k^* always exists. (If $\alpha = 1$, $A + 1 - \delta > 1$ from the assumption and then k^* is well-defined.). Then there exists a feasible allocation $\left\{\bar{c}_t, \bar{k}_t, \bar{h}_t\right\}_{t=0}^{\infty}$ with $\left(\bar{k}_0, \bar{h}_0\right) = (k, h)$ which satisfies $\bar{c}_t = \bar{c} \equiv F(k^*) - k^* > 0$ for all $t \ge 1$. Since $\bar{c}_t$ is time independent

[4] It is sufficient to show that the utility function is not quasi-concave. (Concave function must be quasi-concave.) A set $S_a = \{(c,h) : u(c,h) \ge -a\}$ satisfies

$$S_a = \left\{(c,h) : h \le [(\rho-1)\,a]^{1/\gamma(\rho-1)}\, c^{1/\gamma}\right\}$$

where $a > 0$. Since $1/\gamma > 1$, S is not convex and then u is not quasi-concave.

constant, $\lim_{t\to\infty}\bar{h}_t = \bar{c}$ and $\lim_{t\to\infty} u\left(\bar{c}_t, \bar{h}_t\right) = (1-\rho)^{-1}\bar{c}^{(1-\gamma)(1-\rho)}$. Hence this allocation satisfies $-\infty < \sum_{t=0}^{\infty}\beta^t u\left(\bar{c}_t, \bar{h}_t\right)$. These facts imply:

$$V(k,h) \in (-\infty, 0) \tag{9}$$

Third, we will show that $V(k,h)$ is a nondecreasing function with respect to k and a nonincreasing function with respect to h. Take a feasible allocation $\{c_t, k_t, h_t\}_{t=0}^{\infty}$ with $(k_0, h_0) = (k, h)$. For $\{c_t, k_t, h_t\}_{t=0}^{\infty}$, the allocation $\{c'_t, k'_t, h'_t\}_{t=0}^{\infty}$ with $(k'_0, h'_0) = (k + dk, h)$ such that $dk > 0$ and $c'_t = c_t$ for all t is also feasible. This means that $V(k + dk, h) \geq V(k, h)$. Furthermore, the allocation $\{c'_t, k'_t, h'_t\}_{t=0}^{\infty}$ with $(k'_0, h'_0) = (k, h - dh)$ such that $dh > 0$ and $c'_t = c_t$ for all t is also feasible and this allocation satisfies $h'_t \leq h_t$ for all t. Since $\partial u/\partial h < 0$, this allocation satisfies $\sum_{t=0}^{\infty}\beta^t u(c_t, h_t) \leq \sum_{t=0}^{\infty}\beta^t u(c'_t, h'_t)$ and then $V(k, h - dh) \geq V(k, h)$.

Finally, using these basic properties, we can show the following lemma on continuity of the value function. The continuity of the value function is important for showing the existence of the optimal path.

Lemma 1 *$V(k,h)$ is continuous on $(k,h) \in \mathbb{R}^2_{++}$.*

Proof. In Appendix. ■

2.3 Bellman equation

Using the properties we showed in the last section, we can show that the value function satisfies the Bellman equation. We will construct the optimal allocation later by using this equation.

At first we show that $V(k,h)$ satisfies:

$$V(k,h) = \sup_{c\in(0,F(k))} \{u(c,h) + \beta V(F(k) - c, (1-\lambda)h + \lambda c)\} \tag{10}$$

Since $(F(k) - c, (1-\lambda)h + \lambda c) \in \mathbb{R}^2_{++}$ for $(k,h) \in \mathbb{R}^2_{++}$ and c satisfies $c \in (0, F(k))$, the inside of the bracket at the right hand side of the Bellman operator (10)is well-defined.

First suppose:

$$V(k,h) > \sup_{c\in(0,F(k))} \{u(c,h) + \beta V(F(k) - c, (1-\lambda)h + \lambda c)\}$$

for some (k,h). In this case, some feasible allocation $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ with $(k_0^*, h_0^*) = (k,h)$ satisfies $F(k) - c_0^* > 0$ and:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*, h_t^*) > u(c_0^*, h) + \beta V(F(k) - c_0^*, (1-\lambda)h + \lambda c_0^*) \tag{11}$$

This means $\sum_{t=1}^{\infty} \beta^{t-1} u(c_t^*, h_t^*) > V(F(k) - c_0^*, h_1^*)$, which establishes the contradiction. Next suppose:

$$V(k,h) < \sup_{c\in(0,F(k))} \{u(c,h) + \beta V(F(k) - c, (1-\lambda)h + \lambda c)\} \tag{12}$$

In this case, for some c^*, $V(k,h) < u(c^*,h) + \beta V(F(k) - c^*, (1-\lambda)h + \lambda c^*)$. This implies that, for some allocation $\{c_t, k_t, h_t\}_{t=0}^{\infty}$ with $(k_0, h_0) = (k,h)$ and $c_0 = c^*$, $V(k,h) < \sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$. This also establishes the contradiction.

Finally let us show that there exists $c \in (0, F(k))$ which attains the supremum of (10). In other words, we can use "maximum" instead of "supremum" in the Bellman operator. Define:

$$\bar{V}_{k,h}(c) = u(c,h) + \beta V(F(k) - c, (1-\lambda)h + \lambda c) \tag{13}$$

where k and h are constant. $\bar{V}_{k,h}$ is continuous on $c \in (0, F(k))$ and it satisfies $\lim_{c\to 0} \bar{V}_{k,h}(c) = \lim_{c\to F(k)} \bar{V}_{k,h}(c) = -\infty$. It follows that $\bar{V}_{k,h}(c)$ has a maximum in $c \in (0, F(k))$. Hence $V(k,h)$ satisfies:

$$V(k,h) = \max_{c\in(0,F(k))} \{u(c,h) + \beta V(F(k) - c, (1-\lambda)h + \lambda c)\} \tag{14}$$

2.4 Construction of the optimal interior path

Using the Bellman equation (14), we can show the existence of the interior optimal path. We prove the existence by constructing the optimal allocation $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ from the following process. (A similar method is explained in Stokey and Lucas (1989).)

First, the initial capital stock and habit stock are given and let $k_0^* = k$ and $h_0^* = h$.

Next, suppose the values of k_t^* and h_t^* are determined for some $t = s \geq 0$. Let us determine c_s^*, k_{s+1}^* and h_{s+1}^* by:

$$\begin{aligned}
c_s^* &\in \operatorname*{argmax}_{c\in(0,F(k_s^*))} \{u(c,h_s^*) + \beta V(F(k_s^*) - c, (1-\lambda)h_s^* + \lambda c)\} && (15)\\
k_{s+1}^* &= F(k_s^*) - c_s^* && (16)\\
h_{s+1}^* &= (1-\lambda)h_s^* + \lambda c_s^* && (17)
\end{aligned}$$

Clearly $c_s^* \in (0, F(k_s^*))$, $k_{s+1}^* > 0$ and $h_{s+1}^* > 0$ for all s.

Continuing the process, we can construct the feasible allocation $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ with $(k_0^*, h_0^*) = (k,h)$. This allocation satisfies:

$$\begin{aligned}
V(k,h) &= \left[\sum_{t=0}^{\infty} \beta^t u(c_t^*, h_t^*)\right] + \lim_{t\to\infty} \beta^{t+1} V\left(k_{t+1}^*, h_{t+1}^*\right) && (18)\\
&\leq \sum_{t=0}^{\infty} \beta^t u(c_t^*, h_t^*)
\end{aligned}$$

where the last inequality comes from the fact that $V \leq 0$. Furthermore, since $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ is feasible, $V(k,h) \geq \sum_{t=0}^{\infty} \beta^t u(c_t^*, h_t^*)$. It follows that $V(k,h) = \sum_{t=0}^{\infty} \beta^t u(c_t^*, h_t^*)$. From the above process, it is clear that $c_t^* \in (0, F(k_t^*))$, $k_{t+1}^* > 0$ and $h_{t+1}^* > 0$ for all t.

These results are summarized by the following proposition.

Proposition 2 *There exists a feasible allocation* $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ *with* $(k_0^*, h_0^*) = (k,h) \in \mathbb{R}_{++}^2$ *that maximizes* $\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)$. *In other words,* V *satisfies:*

$$V(k,h) = \max\left\{\sum_{t=0}^{\infty} \beta^t u(c_t, h_t)\right\} \tag{19}$$

subject to $(2), (4), (5)$ *and* (6). *Furthermore, the optimal allocation* $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ *satisfies* $k_t^* > 0$, $c_t^* \in (0, F(k_t^*))$ *and* $h_t^* > 0$ *for all* t *and then the optimal consumption path is interior.*

2.5 First order conditions

Finally let us derive the first order conditions which the optimal path has to satisfy. The interior optimal path satisfies the following the first order conditions. The utility function is continuously differentiable and we have already shown that the optimal path is interior. Hence the first order conditions must hold even if the utility function is nonconcave. Lagrangian is given by:

$$L = \sum_{t=0}^{\infty} \beta^t \{u(c_t, h_t) + \nu_t (F(k_t) - c_t - k_{t+1}) + \varphi_t (h_{t+1} - (1-\lambda) h_t - \lambda c_t)\} \tag{20}$$

where ν_t and φ_t are the lagrange multipliers. The first order conditions with respect to c_t, k_{t+1}, and h_{t+1} are given by:

$$c_t \quad : \quad u_1(c_t, h_t) = \nu_t + \lambda\varphi_t \tag{21}$$

$$k_{t+1} \quad : \quad \nu_t = \beta F'(k_{t+1})\,\nu_{t+1} \tag{22}$$

$$h_{t+1} \quad : \quad \varphi_t + \beta u_2(c_{t+1}, h_{t+1}) - \beta(1-\lambda)\,\varphi_{t+1} = 0 \tag{23}$$

where $t \geq 0$. From (21) and (23), we obtain:

$$u_1(c_t, h_t) - \nu_t + \lambda\beta u_2(c_{t+1}, h_{t+1}) - \beta(1-\lambda)\{u_1(c_{t+1}, h_{t+1}) - \nu_{t+1}\} = 0 \tag{24}$$

Hence the optimal path satisfies:

$$\begin{aligned} & \{F'(k_{t+2}) - (1-\lambda)\}\left\{u_1^t + \lambda\beta u_2^{t+1} - \beta(1-\lambda)\,u_1^{t+1}\right\} \\ = \quad & \beta F'(k_{t+2})\{F'(k_{t+1}) - (1-\lambda)\}\left\{u_1^{t+1} + \lambda\beta u_2^{t+2} - \beta(1-\lambda)\,u_1^{t+2}\right\} \end{aligned} \tag{25}$$

where $u_i^t = u_i(c_t, h_t)$.

3 Transversality conditions

Here we show several proposition on transversality condition. First of all, the transversality condition with respect to k_t is required for the optimal path. Next, we derive some terminal condition with respect to h_t which is very similar to the transversality condition assumed in Francisco et al. (2004).

3.1 Condition on capital stock

Here we will show that the transversality condition on k_t is necessary condition by using a method in Kamihigashi (2002).

Take the optimal allocation $\{c_t^*, k_t^*, h_t^*\}_{t=0}^{\infty}$ with $(k_0^*, h_0^*) = (k, h)$. For such sequence, the allocation $\{c_t, k_t, h_t\}_{t=0}^{\infty}$ with $(k_0, h_0) = (k, h)$ such that:

$$
\begin{aligned}
(c_t, k_t) &= (c_t^*, k_t^*), \quad t \leq T-1 && (26)\\
(c_T, k_T) &= \left(F(k_T^*) - \theta k_{T+1}^*, k_T^*\right) && (27)\\
(c_{T+1}, k_{T+1}) &= \left(\theta c_T^*, \theta k_{T+1}^*\right) && (28)\\
(c_t, k_t) &= (\theta c_t^*, \theta k_t^*), \quad t \geq T+1 && (29)
\end{aligned}
$$

is also feasible, where $\theta \in (0, 1]$. (We can recursively show that $k_t \geq \theta k_t^*$ for such sequence.)

For such sequence,

$$
\begin{aligned}
0 &\leq u\left(F(k_T^*) - \theta k_{T+1}^*, h_T^*\right) - u\left(F(k_T^*) - k_{T+1}^*, h_T^*\right) && (30)\\
&\leq \sum_{t=T+1}^{\infty} \beta^t \{u(c_t^*, h_t^*) - u(c_t, h_t)\}\\
&\leq \sum_{t=T+1}^{\infty} \beta^t \{u(c_t^*, h_t^*) - u(\theta c_t^*, h_t^*)\}\\
&= \left(1 - \theta^{1-\rho}\right) \sum_{t=T+1}^{\infty} \beta^t u(c_t^*, h_t^*) && (31)
\end{aligned}
$$

because $h_t \leq h_t^*$ and $u(\theta c_t^*, h_t^*) = \theta^{1-\rho} u(c_t^*, h_t^*)$. It follows that:

$$
\begin{aligned}
0 &\leq \beta^T \left[\frac{u\left(F(k_T^*) - \theta k_{T+1}^*, h_T^*\right) - u\left(F(k_T^*) - k_{T+1}^*, h_T^*\right)}{1-\theta}\right] && (32)\\
&\leq \frac{1-\theta^{1-\rho}}{1-\theta} \left[\sum_{t=T+1}^{\infty} \beta^t u(c_t^*, h_t^*)\right] && (33)
\end{aligned}
$$

This implies:

$$
\begin{aligned}
0 &\leq \beta^T \lim_{\theta \to 1} \left[\frac{u\left(F\left(k_T^*\right) - \theta k_{T+1}^*, h_T^*\right) - u\left(F\left(k_T^*\right) - k_{T+1}^*, h_T^*\right)}{1-\theta} \right] && (34) \\
&= \beta^T u_1 \left(F\left(k_T^*\right) - k_{T+1}^*, h_T^*\right) k_{T+1}^* && (35) \\
&\leq (1-\rho) \left[\sum_{t=T+1}^{\infty} \beta^t u\left(c_t^*, h_t^*\right) \right] && (36)
\end{aligned}
$$

Since $\lim_{t\to\infty} \sum_{t=T+1}^{\infty} \beta^t u\left(c_t^*, h_t^*\right) = 0$, we can conclude that:

$$
\begin{aligned}
& \lim_{T\to\infty} \beta^T u_1 \left(c_T^*, h_T^*\right) k_{T+1}^* && (37) \\
= & \lim_{T\to\infty} \beta^T u_1 \left(c_T^*, h_T^*\right) \left(A\left(k_T^*\right)^{\alpha} + (1-\delta) k_T^* - c_T^*\right) && (38) \\
= & \ 0 && (39)
\end{aligned}
$$

Furthermore, since $u_1 (c_t, h_t) c_t = (1-\rho) u (c_t, h_t)$, $\lim_{t\to\infty} \beta^t u_1 \left(c_t^*, h_t^*\right) c_t^* = 0$. This implies:

$$
\lim_{T\to\infty} \beta^T u_1 \left(c_T^*, h_T^*\right) k_T^* = 0 \qquad (40)
$$

This condition is called as a transversality condition defined in Stokey and Lucas (1989).

3.2 Condition on habit stocks

Next let us show the following proposition on the terminal condition of habit stocks and consumption.

First, using the first order conditions, we obtain:

$$
\beta^T u_1 \left(c_T^*, h_T^*\right) c_T^* = \beta^T \nu_T c_T^* + \lambda \beta^T \varphi_T c_T^* \qquad (41)
$$

Since $\lim_{T\to\infty} \beta^T u_1 (c_T, h_T) c_T = 0$. Lagrange multipliers are nonnegative and then:

$$\lambda \lim_{T\to\infty} \beta^T \varphi_T c_T^* = \lim_{T\to\infty} \beta^T \nu_T c_T^* = 0 \tag{42}$$

Furthermore, using (21), we obtain:

$$\beta^T \varphi_T \left(\mu h_T^* + \lambda c_T^*\right) + \beta^{T+1} u_2\left(c_{T+1}^*, h_{T+1}^*\right) h_{T+1}^* - \beta^{T+1} \mu \varphi_{T+1} h_{T+1}^* = 0 \tag{43}$$

Since $u_2\left(c_T, h_T\right) h_T = -\gamma\left(1-\rho\right) u\left(c_T, h_T\right)$, $\lim_{T\to\infty} \beta^{T+1} u_2\left(c_{T+1}, h_{T+1}\right) h_{T+1} = 0$. This achieves:

$$\lim_{T\to\infty} \left\{\beta^T \varphi_T h_T^* - \beta^{T+1} \varphi_{T+1} h_{T+1}^*\right\} = 0 \tag{44}$$

Notice that the above condition is weaker than the condition $\lim_{T\to\infty} \left\{\beta^T \varphi_T h_T^*\right\} = 0$.

4 Convergence to the stationary growth path

In this section, we prove that when $\alpha = 1$ and then the production function is $y = Ak$, the optimal path converges to the stationary growth path with the growth rate $\left(\bar{A}\beta\right)^{1/\rho-\gamma(\rho-1)}$ under the following condition on parameters:

[**Assumption**] Parameters λ, β, ρ and γ satisfy:

$$1 - \lambda < \beta^{1/(\rho-1)(1-\gamma)} \tag{45}$$

As we will explain later, this condition assures that the optimal consumption growth rate c_{t+1}/c_t exceeds μ.[5]

[5] If we do not put this assumption, the consumption growth rate which satisfies the first order conditions may be lower than μ. In that case, a growth rate of habit stock is different from the consumption growth rate, because h_t satisfies $h_{t+1}/h_t = \mu + \lambda c_t/h_t \geq \mu$. It is not easy to check whether such a path is optimal or not.

At first we consider the case where $\bar{A}\beta = 1$ and then we will extend the result to the general case.

If $\alpha = 1$, the equation can be written by:

$$\begin{aligned} & u_1(c_t, h_t) + \lambda\beta u_2(c_{t+1}, h_{t+1}) - \beta(1-\lambda) u_1(c_{t+1}, h_{t+1}) \quad (46) \\ = \; & \bar{A}\beta \{u_1(c_{t+1}, h_{t+1}) + \lambda\beta u_2(c_{t+2}, h_{t+2}) - \beta(1-\lambda) u_1(c_{t+2}, h_{t+2})\} \end{aligned}$$

where $\bar{A} = A + 1 - \delta$. This equation appears in Carroll(2000). Let us call the first order condition as the Euler equation.

4.1 Optimal path in the case with $\bar{A}\beta = 1$

If $\bar{A}\beta = 1$, the Euler equation (46) is given by:

$$u_1^t + \lambda\beta u_2^{t+1} - \beta(1-\lambda) u_1^{t+1} = u_1^{t+1} + \lambda\beta u_2^{t+2} - \beta(1-\lambda) u_1^{t+2} \quad (47)$$

(47) means that the value of $u_1^t + \lambda\beta u_2^{t+1} - \beta(1-\lambda) u_1^{t+1}$ is independent of time t. Since $u_1(c,h) = c^{-\rho} h^{-\gamma(1-\rho)}$ and $u_2(c,h) = -\gamma c^{1-\rho} h^{-\gamma(1-\rho)-1}$ we obtain:

$$c_t^{-\rho} h_t^{\gamma(\rho-1)} - \gamma\beta\lambda c_{t+1}^{-(\rho-1)} h_{t+1}^{\gamma(\rho-1)-1} - \beta(1-\lambda) c_{t+1}^{-\rho} h_{t+1}^{\gamma(\rho-1)} = \theta_0, \quad t \geq 0 \quad (48)$$

where θ_0 is a constant. Now let $\mu = 1 - \lambda$. μ satisfies $\mu \in [0, 1)$. Since $c_t = \lambda^{-1}(h_{t+1} - \mu h_t)$, we can express (47) only by habit stocks:

$$\frac{h_t^{\gamma(\rho-1)}}{(h_{t+1} - \mu h_t)^{\rho}} = \frac{\gamma\beta h_{t+1}^{\gamma(\rho-1)-1}}{(h_{t+2} - \mu h_{t+1})^{\rho-1}} + \frac{\beta\mu h_{t+1}^{\gamma(\rho-1)}}{(h_{t+2} - \mu h_{t+1})^{\rho}} + \theta \quad (49)$$

where $\theta = \theta_0/\lambda^{\rho}$ is a constant. We obtain the following result on the sign of θ.

Lemma 3 *Suppose $\bar{A}\beta = 1$. Under the assumption* (45), *$\theta > 0$ where θ is* *defined in* (49).

Proof. See Appendix. ■

Let $\tau = \rho - \gamma(\rho - 1)$ and $\varphi = \gamma\lambda\beta + \beta\mu$. If $\theta > 0$, the difference equation of habit stocks (49) has a unique stationary point:

$$h_\theta^* = \left(\frac{1-\varphi}{\theta}\right)^{1/\tau} \tag{50}$$

Since $1 - \varphi > 0$, h_θ^* is always well-defined.

Now we will show several lemmas that is required for showing that the optimal habit stock path $\{h_t\}_{t=0}^{\infty}$ is monotone.

Lemma 4 *Suppose $\{h_t\}_{t=0}^{\infty}$ is the optimal path. If $h_s \geq h_{s+1}$ and $h_\theta^* \geq h_{s+1}$ for some $s \geq 0$, then $h_t \geq h_{t+1}$ for all $t \geq s+1$.*

Proof. For such s, we can show that $h_{s+1} \geq h_{s+2}$. Since the right hand side of (49) is increasing function with respect to h_{t+2}, $h_{s+1} \geq h_{s+2}$ if and only if:

$$\begin{aligned} & h_s^{\gamma(\rho-1)}\left(h_{s+1} - \mu h_s\right)^{-\rho} \qquad (51)\\ = \; & \gamma\beta h_{s+1}^{\gamma(\rho-1)-1}\left(h_{s+2} - \mu h_{s+1}\right)^{-(\rho-1)} + \beta\mu h_{s+1}^{\gamma(\rho-1)}\left(h_{s+2} - \mu h_{s+1}\right)^{-\rho} + \theta \\ \geq \; & \gamma\beta h_{s+1}^{\gamma(\rho-1)-1}\left(h_{s+1} - \mu h_{s+1}\right)^{-(\rho-1)} + \beta\mu h_{s+1}^{\gamma(\rho-1)}\left(h_{s+1} - \mu h_{s+1}\right)^{-\rho} + \theta \\ = \; & \varphi\lambda^{-\rho} h_{s+1}^{-\tau} + \theta \end{aligned}$$

On the other hand, $\varphi\lambda^{-\rho}h_{s+1}^{-\tau} + \theta \leq \lambda^{-\rho}h_{s+1}^{-\tau}$ because $h_\theta^* \geq h_{s+1}$ and $\varphi\left(h_\theta^*\right)^{-\tau} + \theta = \lambda^{-\rho}\left(h_\theta^*\right)^{-\tau}$ Then the following inequalities:

$$\begin{aligned} \varphi\lambda^{-\rho}h_{s+1}^{-\tau} + \theta & \leq \lambda^{-\rho}h_{s+1}^{-\tau} \qquad (52)\\ & = \left(h_{s+1} - \mu h_{s+1}\right)^{-\rho} h_{s+1}^{-\tau} \\ & \leq \left(h_{s+1} - \mu h_s\right)^{-\rho} h_s^{\gamma(\rho-1)} \end{aligned}$$

always hold, where the last inequality is satisfied because a function $f(x) = (h - \mu x)^{-\rho} x^{\gamma(\rho-1)}$ is increasing and $h_s \geq h_{s+1}$. Hence we can conclude that (51) is always satisfied and then $h_{s+1} \geq h_{s+2}$. Since $h_{s+1} \geq h_{s+2}$ and $h_\theta^* \geq h_{s+2}$, we can also show that $h_{s+2} \geq h_{s+3}$. It follows that $h_t \geq h_{t+1}$, for all $t \geq s$. ■

Lemma 5 *Suppose* $\{h_t\}_{t=0}^\infty$ *is the optimal path. There is no* $s \geq 0$ *such that* $h_s \leq h_{s+1}$ *and* $h_\theta^* < h_{s+1}$.

Proof. In Appendix. ■

Lemma 6 *Suppose* $\{h_t\}_{t=0}^\infty$ *is the optimal path. There does not exist* $t_0 \geq 0$ *such that* $h_t \geq h_{t+1}$ *for all* $t \geq t_0$ *and* $\lim_{t\to\infty} h_t < h^*$.

Proof. In Appendix. ■

Consequently, feasible allocations along the Euler path (49) satisfies the following proposition.

Proposition 7 *Suppose* $\bar{A}\beta = 1$. *Then the optimal habit stock path satisfies one of the following properties for some strictly positive constant* h^*.

[1] $h_t \geq h_{t+1} \geq h^*$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} h_t = h^*$.

[2] $h_t \leq h_{t+1} \leq h^*$ *for all* $t \geq 0$ *and* $\lim_{t\to\infty} h_t = h^*$.

Furthermore, the optimal allocation satisfies:

$$\lim_{t\to\infty} h_t = \lim_{t\to\infty} c_t = h^* \tag{53}$$

Proof. Monotonicity comes from the above lemmas. Since $c_t = \lambda^{-1}(h_{t+1} - \mu h_t)$ and $\lim_{t\to\infty} h_t = h^*$, $\lim_{t\to\infty} c_t = h^*$. ■

4.2 Optimal path for arbitrary A and β

Finally we will derive the property of the optimal consumption growth path in the case where A and β are arbitrary. As we showed above, at the optimal path, $u_1^t + \lambda\beta u_2^{t+1} - \beta\mu u_1^{t+1} > 0$ under the assumption (45). The Euler equation implies:

$$\frac{h_t^{\gamma(\rho-1)}}{(h_{t+1} - \mu h_t)^{\rho}} = \frac{\gamma\beta h_{t+1}^{\gamma(\rho-1)-1}}{(h_{t+2} - \mu h_{t+1})^{\rho-1}} + \frac{\beta\mu h_{t+1}^{\gamma(\rho-1)}}{(h_{t+2} - \mu h_{t+1})^{\rho}} + \phi_0 \left(\bar{A}\beta\right)^{-t} \tag{54}$$

where $\phi_0 > 0$ is a constant. Define $\bar{h}_t = \sigma^{-t} h_t$ where $\sigma = \left(\bar{A}\beta\right)^{1/\tau}$. Substituting $h_t = \sigma^t \bar{h}_t$ into (54) yields:

$$\frac{\bar{h}_t^{\gamma(\rho-1)}}{\left(\sigma\bar{h}_{t+1} - \mu\bar{h}_t\right)^{\rho}} = \frac{\gamma}{\bar{A}} \frac{\bar{h}_{t+1}^{\gamma(\rho-1)-1}}{\left(\sigma\bar{h}_{t+2} - \mu\bar{h}_{t+1}\right)^{(\rho-1)}} + \frac{\mu}{\bar{A}} \frac{\bar{h}_{t+1}^{\gamma(\rho-1)}}{\left(\sigma\bar{h}_{t+2} - \mu\bar{h}_{t+1}\right)^{\rho}} + \phi \tag{55}$$

where $\phi > 0$. This difference equation has one stationary point $\bar{h}_\phi^* > 0$ which satisfies:

$$\bar{h}_\phi^* = \left[\frac{1 - \gamma(\sigma - \mu)/\bar{A} - \mu/\bar{A}}{\phi(\sigma - \mu)^{\rho}}\right]^{1/\tau} \tag{56}$$

Since $\bar{A} \geq 1$, $1 - \gamma(\sigma - \mu)/\bar{A} - \mu/\bar{A}$ and then the value is always well-defined.

Now let us prove that $\lim_{t\to\infty} \bar{h}_t$ exists and $\lim_{t\to\infty} \bar{h}_t > 0$. First suppose $\lim_{t\to\infty} \bar{h}_t = 0$. Define $\bar{g}_t = \frac{\bar{h}_{t+1}}{\bar{h}_t} = \sigma^{-1} g_t$. $\bar{g}_t$ satisfies:

$$\bar{g}_t^{\tau} (\sigma\bar{g}_t - \mu)^{-\rho} = \frac{\gamma}{\bar{A}} (\sigma\bar{g}_{t+1} - \mu)^{-(\rho-1)} + \frac{\mu}{\bar{A}} (\sigma\bar{g}_{t+1} - \mu)^{-\rho} + \phi\bar{h}_{t+1}^{\tau} \tag{57}$$

As we showed previously, under the assumption (45) , $\lim_{t\to\infty} \bar{g}_t = \bar{g}^*$ exists and it satisfies:

$$\sigma\bar{g}^* > \mu \tag{58}$$

$$\bar{g}^{*(\tau-1)} = \frac{\sigma\gamma}{\bar{A}} + \frac{\mu}{\bar{A}} (1 - \gamma) (\bar{g}^*)^{-1} \tag{59}$$

where $\tau = \rho - \gamma(\rho - 1)$. This implies that:

$$\begin{aligned}
\lim_{t\to\infty}\left\{\frac{\beta^{t+1}u(c_{t+1},h_{t+1})}{\beta^t u(c_t,h_t)}\right\} &= \beta \lim_{t\to\infty}\left\{\left(\frac{\sigma\bar{g}_t \ -\mu}{\sigma\bar{g}_{t+1} \ -\mu}\right)^{\rho-1}(\sigma\bar{g}_t)^{-(\tau-1)}\right\} \qquad (60)\\
&= \left(\gamma + \mu(1-\gamma)(\sigma\bar{g}^*)^{-1}\right)^{-1}\\
&> 1
\end{aligned}$$

where the last inequality holds because $\bar{g}^* > \mu/\sigma$, which contradicts the optimality.

Hence the optimal allocation satisfies $\lim_{t\to\infty}\bar{h}_t = \bar{h}^*_\phi > 0$. This implies:

$$\lim_{t\to\infty}\frac{h_{t+1}}{h_t} = \lim\frac{\sigma^{t+1}\bar{h}_{t+1}}{\sigma^t\bar{h}_t} = \sigma \qquad (61)$$

In other words, the growth rate of habit stock approaches σ. Since $\sigma > \mu$ under the assumption (45), the optimal consumption growth rate also approaches σ. ($\lim_{t\to\infty}(h_t/c_t) = \lambda^{-1}[\lim_{t\to\infty}(h_{t+1}/h_t) - \mu] = \lambda^{-1}(\sigma-\mu)$.) This shows the following proposition.

Proposition 8 *Under the assumption* (45)*, the optimal consumption path converges to stationary growth path with the growth rate* $\sigma = \left(\bar{A}\beta\right)^{1/\rho-\gamma(\rho-1)}$.

5 Conclusions

In a discrete time neoclassical growth model with multiplicative habit which is similar to Carroll et al. (2000), we derived the conditions under which the optimal consumption path exists and satisfies the Euler equation. We also derived the convergence of the optimal path to stationary growth path. Especially, the optimal habit stocks is monotone, ruling out oscillatory path. These findings help to put habit formation models on a more secure theoretical foundation.

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APPENDIX: PROOFS

Proof of Lemma 1. Before proving the lemma, let us show that V satisfies:

$$\theta^{-(\rho-1)}V(k,h) \leq V(\theta k,h) \leq V(k,h) \tag{62}$$

$$V(k,h) \leq V(k,\theta h) \leq \theta^{(1-\gamma)(\rho-1)}V(k,h) \tag{63}$$

where $\theta \in (0,1]$. For every feasible allocation $\{c_t,k_t,h_t\}_{t=0}^{\infty}$ with $(k_0,h_0) = (k,h)$, the allocation $\{c'_t,k'_t,h'_t\}_{t=0}^{\infty}$ with $(k'_0,h'_0) = (\theta k,h)$ such that $c'_t = \theta c_t$ for all t is also feasible. For this allocation, h'_t clearly satisfies $h'_t \leq h_t$. It follows that $\theta^{-(\rho-1)}u(c_t,h_t) \leq u(c'_t,h'_t)$.

Next, for every feasible allocation $\{c_t,k_t,h_t\}_{t=0}^{\infty}$ with $(k_0,h_0) = (k,\theta h)$, the allocation $\{c''_t,k''_t,h''_t\}_{t=0}^{\infty}$ with $(k''_0,h''_0) = (k,h)$ such that $c''_t = c_t$ for all t is also feasible. Since $h_t \geq \theta h''_t$ for all t, the allocation satisfies:

$$u(c_t,h_t) \leq \theta^{(1-\gamma)(\rho-1)}u(c''_t,h''_t) \tag{64}$$

These results mean the above inequalities.

Now take a sequence $\{dk_n,dh_n\}_{n=0}^{\infty}$ such that $(k+dk_n,h+dh_n) \in \mathbb{R}^2_{++}$ and $\lim_{n\to\infty}(dk_n,dh_n) = (0,0)$.

$$\begin{aligned} & V(k+dk_n,h+dh_n) - V(k,h) \qquad (65)\\ \leq\ & V(k+|dk_n|,h-|dh_n|) - V(k,h) \\ \leq\ & (1-|dh_n/h|)^{(1-\gamma)(\rho-1)}V(k+|dk_n|,h) - V(k,h) \\ \leq\ & \left\{(1-|dh_n/h|)^{(1-\gamma)(\rho-1)}(1+|dk_n/k|)^{-(\rho-1)} - 1\right\}V(k,h) \end{aligned}$$

and:

$$\begin{aligned} & V\left(k+dk_n, h+dh_n\right)-V(k,h) \\ \geq\ & V\left(k-\left|dk_n\right|, h+\left|dh_n\right|\right)-V(k,h) \\ \geq\ & \left\{\left(1+\left|dh_n/h\right|\right)^{-(1-\gamma)(\rho-1)}\left(1-\left|dk_n/k\right|\right)^{+(\rho-1)}-1\right\} V(k,h) \end{aligned} \tag{66}$$

Since $V(k,h)>-\infty$, $\lim V\left(k+dk_n, h+dh_n\right)=V(k,h)$. ■

Proof of Lemma 3. First suppose $\theta<0$. Since $\sum_{t=0}^{\infty}\beta^t u\left(c_t,h_t\right)=\sum_{t=0}^{\infty}\beta^t u\left\{\left(h_{t+1}-\mu h_t\right)/\lambda, h_t\right\}$ and then:

$$\frac{\partial}{\partial h_{t+1}}\left\{\sum_{t=0}^{\infty}\beta^t u\left(\frac{h_{t+1}-\mu h_t}{\lambda}, h_t\right)\right\}=\lambda^{-1}\left\{u_1^t+\lambda\beta u_2^{t+1}-\beta\mu u_1^{t+1}\right\}<0 \tag{67}$$

For the optimal allocation $\{c_t,k_t,h_t\}_{t=0}^{\infty}$ with $(k_0,h_0)=(k,h)$, consider a new allocation $\{c_t',k_t',h_t'\}_{t=0}^{\infty}$ with $(k_0',h_0')=(k,h)$ such that:

$$\begin{aligned} h_1' &= h_1-dh \\ h_t' &= h_t,\ \ t\geq 2 \end{aligned}$$

where $dh>0$ is constant. For such an allocation, $\{c_t'\}_{t=0}^{\infty}$ satisfies:

$$c_0' = \lambda^{-1}\left(h_1'-\mu h_0'\right)=c_0-\lambda^{-1}dh \tag{68}$$

$$c_1' = \lambda^{-1}\left(h_2'-\mu h_1'\right)=c_1+\left(\lambda^{-1}-1\right)dh \tag{69}$$

$$c_t' = c_t, \quad \text{for } t\geq 2 \tag{70}$$

For sufficiently small dh, $c_t' > 0$ for all t. Furthermore, $\{k_t'\}_{t=1}^{\infty}$ satisfies:

$$\begin{aligned} k_1' &= \bar{A}k - c_0' && (71) \\ &= \bar{A}k - c_0 + \lambda^{-1}dh && (72) \\ \bar{A}^{-t-1}k_{t+1}' &= \bar{A}k - \sum_{s=0}^{t} \bar{A}^{-s}c_t' && (73) \\ &= \bar{A}k - \left\{\sum_{s=0}^{t} \bar{A}^{-s}c_t\right\} + \left(\lambda^{-1} - \bar{A}^{-1}\left(\lambda^{-1} - 1\right)\right)dh \end{aligned}$$

Since $\bar{A} \geq 1$, $\lambda^{-1} - \bar{A}^{-1}\left(\lambda^{-1} - 1\right) > 0$ and then $k_{t+1}' > 0$ for all $t \geq 0$. This implies that this allocation is feasible. Furthermore, such an allocation satisfies $\sum_{t=0}^{\infty} \beta^t u\left(c_t', h_t'\right) > \sum_{t=0}^{\infty} \beta^t u\left(c_t, h_t\right)$ because:

$$\frac{\partial}{\partial h_1} \sum_{t=0}^{\infty} \beta^t u\left(\frac{h_{t+1} - \mu h_t}{\lambda}, h_t\right) < 0 \tag{74}$$

This achieves the contradiction.

Next suppose $\theta = 0$. Let $g_t = \frac{h_{t+1}}{h_t}$. $g_t > \mu$ because $g_t = \mu + \lambda c_t/h_t$ and $\lambda > 0$. The Euler equation (49) implies that $g_t = \frac{h_{t+1}}{h_t}$ satisfies:

$$g_t^{\tau}\left(g_t - \mu\right)^{-\rho} = \gamma\beta\left(g_{t+1} - \mu\right)^{-(\rho-1)} + \beta\mu\left(g_{t+1} - \mu\right)^{-\rho} \tag{75}$$

where $\tau = \rho - \gamma(\rho - 1)$. If the difference equation (75) has a stationary point g^*, then g^* satisfies:

$$\begin{aligned} \left(g^*\right)^{\tau} &= \gamma\beta g^* + \beta\mu(1-\gamma) && (76) \\ g^* &> \mu && (77) \end{aligned}$$

Since $\tau > 1$ and $\gamma\beta > 0$, clearly the solution of (76) and (77) is unique if it exists.

Now let us show that $\{g_t\}_{t=0}^{\infty}$ converges to the unique stationary point g^* under the assumption (45).

Let $f(x) = x^{\tau} - \gamma\beta x - \beta\mu(1-\gamma)$. Under (45), $f(\mu) = \mu^{\tau} - \beta\mu < 0$. Hence the solution of $f(x) = 0$ satisfies $x > \mu$ and g^* really exists. We can also easily check that $g_t \leq g_{t+1}$ if and only if $g_{t+1} \leq g^*$. Then $\lim_{t\to\infty} g_t = g^*$ and :

$$\begin{aligned}\lim_{t\to\infty} \frac{\beta^{t+1} u(c_{t+1}, h_{t+1})}{\beta^t u(c_t, h_t)} &= \lim_{t\to\infty} \left\{ \beta \left(\frac{g_t - \mu}{g_{t+1} - \mu} \right)^{\rho-1} g_t^{-\tau+1} \right\} \\ &= \left(\gamma + \mu(1-\gamma)(g^*)^{-1} \right)^{-1} \\ &> 1\end{aligned} \tag{78}$$

where the last inequality holds because $g^* > \mu$. Hence $\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) = -\infty$. Since $V(k, h) > -\infty$, such a path is not the optimal path. In other words, $\theta > 0$ has to hold at the optimal path. ■

Proof of Lemma 5. We can easily show that if $h_s \leq h_{s+1}$ and $h_\theta^* < h_{s+1}$ for some s, $h_t \leq h_{t+1}$ for all $t \geq s$. Since (49) can have only one stationary point, $\lim_{t\to\infty} h_t = \infty$. (49) implies:

$$\begin{aligned}&\left(\frac{h_{t+1}}{h_t} - \mu \right)^{-\gamma(\rho-1)} (h_{t+1} - \mu h_t)^{-\tau} \\ = \;& \gamma\beta h_{t+1}^{\gamma(\rho-1)-1} (h_{t+2} - \mu h_{t+1})^{-(\rho-1)} + \beta\mu h_{t+1}^{\gamma(\rho-1)} (h_{t+2} - \mu h_{t+1})^{-\rho} + \theta\end{aligned} \tag{79}$$

Since $g_t \geq 1$ and $\lim_{t\to\infty} h_t = \infty$, the left hand side of (49) approaches 0 as t goes to ∞. This establishes a contradiction. ■

Proof of Lemma 6. Suppose not. In this case, $\lim_{t\to\infty} h_t = 0$. The Euler equation (49) implies:

$$g_t^{\tau} (g_t - \mu)^{-\rho} = \gamma\beta (g_{t+1} - \mu)^{-(\rho-1)} + \beta\mu (g_{t+1} - \mu)^{-\rho} + \theta h_{t+1}^{\tau} \tag{80}$$

Since $\lim_{t\to\infty} h_{t+1}^{\tau} = 0$ and $g_{t+1} \leq 1$, for any $\varepsilon > 0$, there exist $T_\varepsilon \in \mathbb{N}$ such that:

$$\theta h_{t+1}^{\tau} \leq \theta\varepsilon \left(g_{t+1} - \mu\right)^{-\rho} \tag{81}$$

for all $t \geq T_\varepsilon$. This implies that:

$$g_t^{\tau} \left(g_t - \mu\right)^{-\rho} \geq \gamma\beta \left(g_{t+1} - \mu\right)^{-(\rho-1)} + \beta\mu \left(g_{t+1} - \mu\right)^{-\rho} \tag{82}$$

and:

$$g_t^{\tau} \left(g_t - \mu\right)^{-\rho} \leq \gamma\beta \left(g_{t+1} - \mu\right)^{-(\rho-1)} + \left(\beta\mu + \theta\varepsilon\right) \left(g_{t+1} - \mu\right)^{-\rho} \tag{83}$$

for all $t \geq T_\varepsilon$. First of all, for each $\varepsilon \geq 0$, consider a sequence $\{\hat{g}_t(\varepsilon)\}_{t=T_\varepsilon}^{\infty}$ such that $\hat{g}_{T_\varepsilon} = g_{T_\varepsilon}$ and:

$$\hat{g}_t{}^{\tau} \left(\hat{g}_t - \mu\right)^{-\rho} = \gamma\beta \left(\hat{g}_{t+1} - \mu\right)^{-(\rho-1)} + \left(\beta\mu + \theta\varepsilon\right) \left(\hat{g}_{t+1} - \mu\right)^{-\rho} \tag{84}$$

for $t \geq T_\varepsilon + 1$. We can easily see from the difference equation (84) that for fixed $\hat{g}_t$, the value of $\hat{g}_{t+1}$ is uniquely determined and these values satisfy $d\hat{g}_{t+1}/d\hat{g}_t > 0$. Furthermore, $g_{T_\varepsilon+1}$ satisfies:

$$\hat{g}_{T_\varepsilon+1}(0) \geq g_{T_\varepsilon+1} \geq \hat{g}_{T_\varepsilon+1}(\varepsilon)$$

Hence we can recursively prove that:

$$\hat{g}_t(0) \geq g_t \geq \hat{g}_t(\varepsilon) \text{ for all } t \geq T_\varepsilon \tag{85}$$

Now let $f_\varepsilon(x) = x^\tau - \gamma\beta x - (\beta\mu + \theta\varepsilon)(1-\gamma)$. The assumption (45) means $\mu^\tau - \beta\mu < 0$ and then $f_\varepsilon(\mu) = \mu^\tau - \beta\mu - \theta\varepsilon(1-\gamma) < 0$. Hence the solution of $f_\varepsilon(x) = 0$ is unique and satisfies $x > \mu$. It follows that the stationary point of the above difference equation, $\hat{g}^*(\varepsilon)$ really exists and satisfies:

$$\hat{g}^*(\varepsilon)^\tau = \gamma\beta(\hat{g}^*(\varepsilon) \quad - \mu) + (\beta\mu + \theta\varepsilon)$$

We can easily check that $\hat{g}_t(\varepsilon) \leq \hat{g}_{t+1}(\varepsilon)$ if and only if $\hat{g}_{t+1}(\varepsilon) \leq \hat{g}^*(\varepsilon)$. Hence we can conclude that $\lim_{t\to\infty} \hat{g}_{t+1}(\varepsilon) = \hat{g}^*(\varepsilon)$. Using (85), we can conclude that $\limsup g_t \leq \hat{g}^*(0)$ and $\liminf g_t \geq \hat{g}^*(\varepsilon)$. Since ε can be arbitrary small and $\lim_{\varepsilon\to 0} \hat{g}^*(\varepsilon) = \hat{g}^*(0)$,[6] we obtain:

$$\lim_{t\to\infty} g_t = \hat{g}^*(0) > \mu \tag{86}$$

This implies:

$$\lim_{t\to\infty} \frac{\beta^{t+1}u(c_{t+1}, h_{t+1})}{\beta^t u(c_t, h_t)} = \left(\gamma + \mu(1-\gamma)(\hat{g}^*(0))^{-1}\right)^{-1} > 1 \tag{87}$$

where the last inequality holds because $\hat{g}^*(0) > \mu$. Such a sequence has to satisfy $\sum_{t=0}^{\infty} \beta^t u(c_t, h_t) = -\infty$. It follows that such a path is not the optimal path. ■

[6] We can show the equation strictly, but this relationship is obvious graphically.