Private Monitoring and Communication in Cartels: Explaining Recent Collusive Practices*

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Abstract

Motivated by recent cartel practices, a stable collusive agreement is characterized when firms' prices and quantities are private information. Conditions are derived whereby an equilibrium exists in which firms truthfully report their sales and then make transfers within the cartel based on these reports. The properties of this equilibrium fit well with the cartel agreements used in a number of markets including citric acid, lysine, and vitamins.

1 Introduction

1.1 Some Recent Cartels

1.1.1 Lysine

Lysine is added to livestock feed to develop body tissue in pigs and poultry. In the early 1990s, the five major lysine producers formed a global cartel which lasted until mid 1995. Though two of the firms - Ajinomoto and Sewon - recommended colluding through the allocation of exclusive geographic markets, Archer Daniel Midlands pushed for and succeeded in having accepted a sales quota scheme whereby each firm was entitled to a certain level of output. The agreed-upon allocation for 1992 for the

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global and European markets is shown in Table 1.[1]

Table 1 - Lysine Market Allocation (1992, tons)

Company	Global	Europe
Ajinomoto	73,500	34,000
Archer Daniels Midland	48,000	5,000
Kyowa	37,000	8,000
Sewon	20,500	13,500
Cheil	6,000	5,000

To monitor compliance, Kanji Mimoto of Ajinomoto was assigned the task of preparing monthly "scorecards" for the cartel. Each company telephoned or mailed their sales volumes to Mimoto, who then prepared a spreadsheet that was distributed at the quarterly meetings of the cartel. To promote compliance, "guaranteed buy-ins" were used: A company which sold more than its quota had to buy output from producers who were below quota.

There were isolated reports of cartel members under-reporting their sales in order to avoid the punishment associated with guaranteed buy-ins. For example, Cheil claimed to the European Commission that it provided "misleading" sales information to the other companies, while Ajinomoto hid 3,500 tons of lysine from the cartel's auditors; an internal memo read: "Hide 1,000 tons in Thailand internal business." Nevertheless, the collusive agreement was largely successful.

1.1.2 Citric Acid

Citric acid is primarily used in the food and beverage industry, but is also an ingredient in household cleaning products, pharmaceuticals, and cosmetics as well as having some industrial uses. From early 1991 to mid 1995, the five largest producers of citric acid operated a cartel. At the time they made up about 60% of global production and 67% of production in the European Union. A sales quota scheme was established in terms of market shares. Reported in Table 2, these market shares were based on the average of the previous three years' sales.

Table 2 - Citric Acid Market Allocation (1991)

Company	Market Share
Haarman & Reimer	32.0%
Archer Daniels Midland	26.3%
Jungbunzlauer	23.0%
Hoffman LaRoche	13.7%
Cerestar Bioproducts	5.0%

As with the lysine cartel, the market allocation was monitored through the reporting of sales. On a monthly basis, each company reported its sales to an executive of

[1] All ensuing facts in Section 1 are from European Commission decisions and can be found in Harrington (2006).

Hoffmann LaRoche. The data was then assembled and reported back to the members by telephone. To provide some external validity, the reported sales were checked by independent Swiss auditors. Enforcement was through a "buy-back system" whereby a company that exceeded its assigned quota in any one year was obliged to purchase product from the companies with sales below their quota in the following year. For example, at a cartel meeting in November 1991, it was determined that Haarmann & Reimer needed to buy 7,000 tons of citric acid from ADM and it seemed the purchase was later made. In terms of efficacy, actual production by each member adhered very closely to the cartel's planned production.

1.1.3 General Properties of Recent Cartels

There are a number of properties common to the citric acid and lysine cartels that we would like to highlight. First note that demand in these markets came from industrial buyers, with price typically being set bilaterally between a seller and a buyer. This meant that price, along with firm sales, were not public information. Second, the collusive agreement was in terms of an allocation of sales, and not just coordination on a common price. Third, the collusive agreement was monitored by comparing sales to the agreed-upon quotas. Fourth, monitoring used self-reported sales which, on the whole, were not verifiable. Fifth, the collusive agreement was (at least partly) enforced through a transfer scheme whereby firms which reported sales above their quota effectively made a payment (through inter-firm purchases) to those firms who reported sales below their quota.

These properties are not unique to the citric acid and lysine cartels. The setting of sales quotas with monitoring in terms of reported sales was also a practice deployed by cartels in the markets for carbonless paper, choline chloride, copper plumbing tubes, graphite electrodes, plasterboard, vitamins, and zinc phosphate. For example, from the European Commission decision on the vitamins cartel: "The purpose of the quarterly meetings was to monitor achieved market shares against quota and to adjust sales levels to comply with the agreed allocations."[2] And for the graphite electrodes cartel:[3]

> For the purpose of formalising the exchange of volume information and making the collection of data more efficient, SGL proposed at the "Top Guy" meeting in Tokyo in February 1995 the adoption of a "Central Monitoring System". Tokai was designated by the cartel to collect the data from the Japanese producers, UCAR and SGL.

The accuracy of reported sales was established ex post for the carbonless paper cartel: "Comparison of these figures with information on real sales figures confirms

[2] Quoting from the European Commission decision for the vitamin B2 cartel (Harrington, 2006, p. 47).

[3] Quoting from the European Commission decision for the graphite electrodes cartel (Harrington, 2006, p. 48).

that the sales volume information exchanged at the meeting was accurate."[4] Finally, the use of a transfer scheme based upon reported sales was also documented for cartels in choline chloride, organic peroxides, sodium gluconate, sorbates, vitamins, and zinc phosphate. For example, for the choline chloride cartel "... it was understood that Akzo Nobel and UCB could claim 35% and 28% respectively, while BASF would have 15%. The principle was accepted that compensation should be provided if these shares were exceeded."[5]

In fact, these collusive practices are not a recent phenomenon as they were present in the International Steel Agreement of 1926. Collusion was between countries (not companies) and an explicit (though not enforceable) contract was written up, from which we quote.[6] Sales quotas were fixed according to Articles 3 and 4 and resulted in the following allocation:

Table 3 - International Steel Agreement (1926)

Country	Allocated Market Share
Germany	40.45%
France	31.89%
Belgium	12.57%
Luxemburg	8.55%
Saar Territory	6.54%

Article 5 specified that monitoring would be in terms of sales: "Every month each country's actual net production of crude steel during that month shall be ascertained, in relation to the figures indicated by the quotas." And in Articles 6 and 7, penalties were specified in terms of monetary transfers between firms: "If the quarterly production of a country exceeds the quota which was fixed for it, that country shall pay in respect of each ton in excess a fine of 4 dollars, which shall accrue to the common fund. ... If the production of any country has been below the quota allotted to it, that country shall receive in compensation from the common fund the sum of two dollars per ton short." Thus, the collusive practices observed in more recent years have been in use for at least 80 years.

1.2 Research Objective

What the preceding summary of cartels reveals is that, when faced with colluding in an environment in which prices and quantities are not easily observed, firms responded with a similar design to their collusive agreement. Towards better understanding hard core cartels, the primary objective of our research is to explain how these well-documented collusive practices were effective in sustaining collusion. What prevents

[4] Quoting from the European Commission decision for the carbonless paper cartel (Harrington, 2006, p. 51).

[5] Quoting from the European Commission decision for the choline chloride cartel (Harrington, 2006, p. 58).

[6] A copy of this agreement is in Appendix I of Plummer (1938). It is unclear what was the geographic area to which the agreement applied - it may have just been the European continent - since it did not encompass all countries that were producing steel at that time.

firms from undercutting the collusive price and then under-reporting their sales? How can cartel members be induced to truthfully report their sales when higher sales reports require providing compensation to other members? We show that, if market demand is not too volatile, an equilibrium exists in which firms truthfully report their sales and set collusive prices, with asymmetric punishments that could be implemented using guaranteed buy-ins or buy-backs. Thus, observed collusive practices fit quite well within the equilibrium framework.

Having provided an explanation of observed cartel behavior, a second task is to explore the structure of optimal equilibria when prices and quantities are private information. While observed collusive agreements were effective, are there other designs which would generate higher cartel profits? For a simple demand setting, this question is addressed using a mechanism design approach. Optimal equilibria are found to have properties consistent with those of our equilibrium which suggests that these cartels have found a winning design, and thus could explain why it is so widespread.

Our informational setting - in which the history of actions is private information - is not new to the repeated game literature, and various Folk Theorems have been derived. A review of that work is provided in Appendix A where we explain that previous models are either inappropriate for the oligopoly setting and/or characterize an equilibrium that does not conform with collusive practices. While the oligopoly game we examine is not a member of the class of games considered in those papers,[7] the more crucial distinction is in terms of objectives. The focus of previous work on private monitoring is characterizing the set of equilibrium payoffs. In contrast, we want to explain observed collusive agreements by constructing an empirically valid equilibrium which sustains collusion when firms do not publicly observe prices or quantities.[8]

The model is described in Section 2 and, as a benchmark, the static Nash equilibrium is characterized in Section 3. The main result is in Section 4 where we show that, under certain demand conditions, collusion can be sustained even when prices and sales are private information. Section 5 focuses on the special case when market demand can take 0, 1, or 2 units and characterizes an optimal collusive equilibrium. Section 6 concludes.

2 Model

There are $n \geq 2$ firms who, in each period of an infinite horizon setting, simultaneously choose price from a compact set, after which each firm's sales are stochastically determined. (As is typical, it is assumed that a firm supplies to meet demand.) The stochastic realization for period t is composed of total demand, m^t, and an allocation of that demand described by a vector of firms' quantities, $\underline{q}^t$. Market de-

[7] At a minimum, our action space is infinite - firms can choose any non-negative price - while previous work (excluding Aoyagi, 2002) assumes a finite action space.

[8] Our objective is then the same as Graham and Marshall (1987) who seek to explain observed collusive practices in auctions.

mand is integer-valued and lies in the finite set $\Gamma \equiv \{\underline{m}, \underline{m}+1, \ldots, \overline{m}-1, \overline{m}\}$, where $0 \leq \underline{m} < \overline{m}$ and $\overline{m}$ is finite. Let $\rho : \Gamma \rightarrow [0,1]$ be a probability function where $\rho(m)$ is the probability that the total demand is m. Define

$$\mu \equiv \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\, m$$

as the average market sales. Note that market demand does not depend on firms' prices. While firm demand will be sensitive to price, market demand is perfectly inelastic.[9] Firms have a common constant marginal cost of c.

An individual firm's demand has support $\{0, 1, \ldots, \overline{q}\}$ where $\overline{q}$ is finite and, of course, $\overline{q} \leq \overline{m}$. (Assuming the lower bound to firm sales/quantity is zero is not important.) Let $\Psi\left(\underline{q}; m, \underline{p}\right)$ denote the probability that the quantity vector is $\underline{q} = \{q_1, ...q_n\}$, given price vector $\underline{p} = \{p_1, ...p_n\}$ and market demand m. To focus on symmetric equilibria, we assume that the probability distribution Ψ of how the market demand is split is symmetric across the firms (i.e. it is invariant to permutations of firm identities):

A1 $\Psi\left(\underline{q}; m, \underline{p}\right) = \Psi\left(\omega\left(\underline{q}; i, j\right); m, \omega\left(\underline{p}; i, j\right)\right)$ $\forall i, j$, $\forall\left(\underline{q}, \underline{p}\right)$, where $\omega\left(\underline{q}; i, j\right)$ is the vector $\underline{q}$ when elements i and j are exchanged.

Next let $\psi_i\left(q; m, \underline{p}\right)$ be the probability function on firm i's sales given total demand m and the price vector $\underline{p}$ (ψ_i is a marginal distribution of Ψ). Let $\sigma_i\left(\cdot; q, \underline{p}\right)$ denote firm $i's$ beliefs on total sales given it sells q units and the price vector $\underline{p}$. By Bayes' Rule,

$$\sigma_i\left(m; q, \underline{p}\right) = \frac{\rho(m)\, \psi_i\left(q; m, \underline{p}\right)}{\sum_{m'=\underline{m}}^{\overline{m}} \rho(m')\, \psi_i\left(q; m', \underline{p}\right)}.$$

Two conditions are required of σ_i (which implicitly places conditions on ρ and ψ_i). A2 specifies that a firm always assigns positive probability to demand equalling its maximum value, which is weaker than assuming σ_i has full support. A3 assumes that the higher a firm's quantity, the more weight the firm attaches to total demand being higher.[10]

A2 $\sigma_i\left(\overline{m}; q, \underline{p}\right) > 0$, $\forall q$, $\forall \underline{p}$.

A3 If $q' > q''$ then $\sigma_i\left(\cdot; q', \underline{p}\right)$ first-order stochastically dominates (FOSD) $\sigma_i\left(\cdot; q'', \underline{p}\right)$, $\forall q', q''$, $\forall \underline{p}$.

[9] While we make this assumption for purposes of tractability, it is reasonable for many of the cartels mentioned in the Introduction; for example, the choline chloride, citric acid, lysine, and vitamins cartels. These products are inputs in a downstream product and thus their demand is derived demand. Since the input makes up a very small percentage of the downstream product's cost of production, market demand will generally be insensitive to price for a wide range of prices.

[10] A2 and A3 hold, for example, when Ψ is derived from a binomial distribution; that is, each of m customers independently choose from which firm to buy. A proof is available from the authors upon request.

The setting is an infinitely repeated game in which, in each period, firms choose price and then stochastic demand is realized. Let δ be the common discount factor and assume a firm acts to maximize the expected present value of its profit stream. Each firm's price and realized sales are private information. This structure is augmented by allowing firms to make public messages and conduct monetary transfers. The sequence of decisions and events is described below.

Stage 1 (price) Each firm chooses a non-negative price.

Stage 2 (sales) With prices being private information, each firm learns its sales.

Stage 3 (report) With prices and sales being private information, firms simultaneously submit publicly observed costless messages from the set $\{\varnothing, 0, 1, \ldots, \overline{q}\}$ where $\varnothing$ means providing no message. (A message is to be interpreted as a sales report).[11]

Stage 4 (transfer) With prices and quantities being private information and reports being public information, each firm makes a publicly observed non-negative payment which is divided equally among the other $n-1$ firms.

To place this model in perspective, the problem we are tackling is one of collusion with imperfect monitoring, which originated with Stigler (1964) and was first formally treated in a game-theoretic setting by Green and Porter (1984). With the latter treatment, which has become the standard approach, firms choose quantities in a homogeneous goods industry with price being determined by those quantities and an unobserved demand shock. Firms' quantities are private information, while price is a public signal. In contrast, we assume both prices and quantities are private information in the context of a price-setting game. More closely related is our earlier work (Harrington and Skrzypacz, 2007), from which the current paper differs in three ways. First, firms' quantities are private information, which is at the heart of the problem we're addressing here. Second, market demand is stochastic, which is empirically compelling and required to make the problem interesting (this is explained in Section 4). Third, each firm sets a single price for all customers instead of a customer-specific price. This third assumption appears to be more for convenience though a more careful assessment of that claim is needed.[12]

[11] This specification modestly departs from actual cartel practices. For example, the citric acid cartel had Haarman & Reimer, ADM, Jungbunzlauer, and Cerestar simultaneously submit sales reports to Hoffman LaRoche, after which Hoffman LaRoche added its sales report and then disseminated all of the reports. This meant that Hoffman LaRoche submitted its sales report *after* learning the reports of the other four cartel members.

[12] Also related is Marshall and Marx (2008). They assume that any deviation in price is perfectly identified from firms' market shares, while we assume that market shares are subject to shocks and thus there is imperfect monitoring.

3 Static Nash Equilibrium

Before considering the infinitely repeated setting, let's establish the non-collusive benchmark. Firm i's expected profit is

$$\pi_i(p_1, \ldots, p_n) = \sum_{m=\underline{m}}^{\overline{m}} \rho(m)(p_i - c)\underline{q_i} \cdot \psi_i(m, \underline{p}).$$

where $\underline{q_i}$ is a vector $(0, 1, ...\overline{m})$, $\psi_i(m, \underline{p})$ is a vector of $\psi_i(q_i; m, \underline{p})$ for all q_i and "$\cdot$" is the dot product.

Let $p^N(c)$ denote a symmetric Nash equilibrium.

$$p^N(c) \in \arg\max \sum_{m=\underline{m}}^{\overline{m}} \rho(m)(p_i - c) q_i \cdot \psi_i(m, p^N, \ldots, p_i, \ldots, p^N).$$

We will assume the demand structure allows for a symmetric Nash equilibrium for the static game, as specified in Assumption A4:

A4 The static game with cost c has, $\forall c \geq 0$, a symmetric Nash equilibrium price $p^N(c)$ that is increasing, continuous, and unbounded in c.

A sufficient condition for A4 to hold is that: i) $\psi_i(q; m, \underline{p})$ depends only on pair-wise price differences, which can be derived from a model of consumer choice with quasi-linear preferences in money; ii) the first-order condition (FOC) is sufficient to characterize equilibrium for any cost; and iii) $\psi_i(q; m, \underline{p})$ is interior for any equilibrium prices. Indeed, as we show in Appendix B, under these conditions a firm's equilibrium price equals its cost plus a constant, which satisfies A4.

4 Sustaining Collusive Outcomes

Before moving on to the main result, let us first note that sustaining collusion is trivial when market demand is common knowledge, either because it is fixed or is stochastic but observed. For suppose it is common knowledge that market demand is m'. If firm i expects the other firms to make truthful sales reports then firm i knows that if its report is inaccurate then total reported sales will differ from m'. Though the other firms will not know who delivered a misleading sales report, it will be common knowledge among them that someone did. Thus, if there is a (common) punishment when total reported sales differ from m', firms can be induced to report truthfully if they are sufficiently patient. Once firmed are motivated to report truthfully, the collusive scheme presented in Harrington and Skrzypacz (2007) can sustain collusion.[13] We then assume market demand is stochastic and unobserved.

Consider a symmetric strategy profile in which there are two phases: *collusive* and *non-collusive*. Suppose the industry is in the collusive phase in period t. During

[13] This collusive scheme is described in the proof of Theorem 1, as we deploy it here as well.

the price stage, the strategy has a firm set the collusive price. After firm i learns its sales - which is denoted q_i^t - it then truthfully reports those sales. If all firms have submitted reports then firm i pays zr_i^t to the other firms where r_i^t was the firm's sales report in the previous stage (in equilibrium, $r_i^t = q_i^t$). If all firms make their payments then there is a public randomization device which determines whether the industry remains in the collusive phase or shifts to the non-collusive phase. The probability function for shifting to the non-collusive phase is denoted $\phi : \{0, 1, 2, \ldots\} \to [0, 1]$. If some firm does not submit a report or failed to make the appropriate payment then the industry shifts to the non-collusive phase. Once an industry is in the non-collusive phase, firms price according to the static Nash equilibrium, don't make reports, and don't make payments. This strategy profile we refer to as the *lysine strategy profile* and is summarized below for firm i.

- In the price stage:
  - if in the collusive phase then price at $\widehat{p}$
  - if in the non-collusive phase then price at p^N
- In the report stage:
  - if in the collusive phase then report q_i^t
  - if in the non-collusive phase then do not report
- In the transfer stage:
  - if in the collusive phase,
    - and all firms reported then make a payment of $r_i^t z$ (which is then equally divided among the other $n-1$ firms).
    - and one or more firms did not report then make a zero payment
  - if in the non-collusive phase then make a zero payment
- In the transition stage (public randomization device):
  - if in the collusive phase,
    - all firms reported and $x_j^t = r_j^t z \;\forall j$ (where x_j^t is firm j's payment), then firms remain in the collusive phase with probability $1 - \phi\left(\sum_{j=1}^n r_j^t\right)$ and shift to the non-collusive phase with probability $\phi\left(\sum_{j=1}^n r_j^t\right)$
    - otherwise, go to the non-collusive phase with probability one
  - if in the non-collusive phase then remain in the non-collusive phase with probability one.

The lysine strategy profile fits reasonably well recent collusive practices in markets such as citric acid, lysine, and vitamins. For when firms' prices and quantities are private information - an appropriate feature of those markets - the cartel monitors the agreement by having firms report their sales, and then punish for reported over-production by having monetary transfers move from over-producers to under-producers. In practice, these transfers were performed through inter-firm sales which, as long as price exceeds cost, act like monetary transfers. (Unfortunately, we lack documentation regarding the price of inter-firm sales.) A desirable feature of performing transfers in this manner is that they did not create suspicions since inter-firm sales were a common competitive feature of these markets. The one feature of the lysine strategy profile that is not expressly referred to in the documentation of these cases is the conditions under which the cartel collapses. The lysine strategy profile specifies that the probability of collusion stopping depends on firms' sales reports, as well as whether the requisite payments were made. In practice, evidence is scarce that such contingencies are expressly discussed by cartel members; even the 1926 International Steel Agreement did not specify in the contract what would happen if a firm did not pay the required \$4 for each ton above its quota. Our working assumption is that implicit in any collusive agreement is that egregious behavior - not making the requisite payment, incredible sales reports, and the like - risks causing the cartel to collapse.

Theorem 1 provides sufficient conditions whereby the lysine strategy profile is a semi-public perfect equilibrium, which is a sequential equilibrium satisfying certain properties. Defined and used in Compte (1998), this solution concept has actions depend only on the public history, and messages depend only on the public history and the most recent private history.[14] In our setting, this means that prices and payments depend only on the public history of past reports and payments, while a firm's report depends only on the public history of reports and payments and the private information composed of the firm's price and sales in the current period.

Given a condition on the volatility of market demand, Theorem 1 shows that collusive outcomes are sustainable when firms are sufficiently patient. More specifically, if firms are sufficiently patient and (1) holds then a collusive price $\widehat{p}$ is supportable using the lysine strategy profile and, in addition, the probability of transiting to the non-collusive phase is arbitrarily small. In interpreting (1), recall that $\overline{m}$ is the maximal value of market demand and μ is average market demand. As $p^{N^{-1}}(\cdot)$ is the inverse of the static Nash equilibrium price function, then $p^{N^{-1}}(\widehat{p})$ is the marginal cost for which the static Nash equilibrium price equals $\widehat{p}$. Since $\widehat{p} > p^N(c)$, then $p^{N^{-1}}(\widehat{p}) - c > 0$. (1) will hold if $\rho(\cdot)$ puts sufficient mass on maximal demand so that average demand is close to maximal demand.[15] Thus, Theorem 1 shows that if demand is not "too stochastic" - in the restricted sense that average demand is not too much lower than maximal demand - then collusion can be sustained when firms are sufficiently patient. Proofs are in Appendix C.

[14] To be more exact, messages depend on the private history *since the last period in which messages were informative*. Compte (1998) considered equilibria in which there was delay in the sending of informative messages. In the equilibrium we characterize, there is no delay.

[15] Given that the lhs of (1) is unbounded as $\mu \to \overline{m}$ and the rhs is bounded then (1) is satisfied.

Theorem 1 *For any $\widehat{p} > p^N$ and $\varepsilon > 0$, there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ and*

$$\frac{\mu}{\overline{m} - \mu} > \frac{(n-1)\left[p^{N^{-1}}(\widehat{p}) - c\right]}{\widehat{p} - p^N} \tag{1}$$

then the lysine strategy profile is a semi-public perfect equilibrium and

$$max\left\{\phi(m) : \underline{m} \leq m \leq \overline{m}\right\} < \varepsilon.$$

If we impose the additional structure on demand used in Appendix B to deliver $p^N(c) = c + const$ (namely, that consumers have quasi-linear preferences in money) then (1) takes a more easy-to-interpret form:

$$\frac{\mu}{\overline{m} - \mu} > n - 1. \tag{2}$$

In that case, Theorem 1 can be re-stated as:[16]

Corollary 2 *Suppose $p^N(c) = c + const$. If $\frac{\mu}{\overline{m}-\mu} > n - 1$ then for any $\widehat{p} > p^N$ and $\varepsilon > 0$, there exists $\underline{\delta} < 1$ such that if $\delta > \underline{\delta}$ then the lysine strategy profile is a semi-public perfect equilibrium with $max\left\{\phi(m) : \underline{m} \leq m \leq \overline{m}\right\} < \varepsilon$.*

Given firms truthfully report their sales, the mechanism used to sustain a collusive price is the same as in Harrington and Skrzypacz (2007). For each unit that a firm sells (and reports), it makes a payment of z to the other cartel members. Thus, when a firm sells a unit rather than have the unit sold by another firm, it ends up paying z rather than receiving $\frac{z}{n-1}$, so the net effect is $\left(\frac{n}{n-1}\right)z$. This means that a firm's effective marginal cost is $c+\left(\frac{n}{n-1}\right)z$. As a firm's price only impacts its current profit and its net transfer - but not its future payoff - the equilibrium (collusive) price is simply the static Nash equilibrium price when each firm faces a marginal cost of $c + \left(\frac{n}{n-1}\right)z$. A higher collusive price can then be sustained by setting a higher per unit transfer.

The preceding argument rests on a firm truthfully reporting its sales, and there is clearly an incentive to under-report since doing so reduces the required payment to other firms. The collusive mechanism offsets this temptation to under-report by making it more likely that the cartel breaks down when the aggregate sales report is smaller.[17] Specifically, we use the following specification in the proof of Theorem 1:

$$\phi(m) = \begin{cases} \beta(\overline{m} - m)(1-\delta) & \text{if } m \leq \overline{m} \\ (1-\delta)^{\omega} & \text{if } m > \overline{m} \end{cases} \tag{3}$$

[16] In a proof that is available on request, it has been shown that the binomial case satisfies A3 and, therefore, satisfies A1-A4.

[17] While we have specified a punishment of infinite reversion to the stage game Nash equilibrium, this collusive mechanism will also work with finite reversion in which case there are periodic price wars.

where $\omega \in (0,1)$ and[18]

$$\beta \geq \frac{(n-1)\left[p^{N^{-1}}(\widehat{p}) - c\right]}{\delta\left\{\mu\left(\widehat{p} - p^{N}(c)\right) - (n-1)\left[p^{N^{-1}}(\widehat{p}) - c\right](\overline{m} - \mu)\right\}}. \tag{4}$$

This probability of cartel breakdown (see Figure 1) is decreasing and linear for equilibrium values of the aggregate sales report. Thus, under-reporting is discouraged because a lower sales report is more likely to cause collusion to end. However, in counter-acting the incentive to under-report, one could create an incentive to *over-report*; a firm reports higher sales and makes a higher payment in order to reduce the likelihood of cartel collapse. Here, we use the fact that an over-report results in reported market sales taking on, with positive probability, a non-equilibrium value (that is, in excess of maximal demand $\overline{m}$). While the probability of a price war is decreasing in total reported sales for *equilibrium* values of total demand, the probability of a price war is higher when total reported sales exceeds maximal demand (at least when $\delta \to 1$). Firms then do not want to over-report sales either.[19] Finally, as a positive probability of collusion ending is needed to induce firms to truthfully report their sales, there is an inefficiency. However, the probability of cartel breakdown goes to zero as $\delta \to 1$.

To explain the source of the restriction in (1), we need to examine the following incentive compatibility constraint (ICC):

$$\delta\left(V - V^{N}\right)\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m \left| q, \underline{p}\right.\right)\left[\phi(m-1) - \phi(m)\right] \geq z. \tag{5}$$

This ICC ensures it is optimal to report truthfully than to under-report by one unit. Under-reporting by one unit reduces firm 1's payment to other firms by z, which is the rhs of (5). The lhs is the expected reduction in the future payoff from reporting $q-1$ rather than q. The probability of cartel breakdown is increased by $\phi(m-1) - \phi(m)$ (when the market demand realization is m) and the foregone future payoff due to cartel breakdown is $V - V^{N}$ where V is the collusive value and V^{N} is the non-collusive value. In the proof of Theorem 1, it is shown that (5) is the binding ICC and is the source of (1).[20]

To show how (1) implies (5), first note that

[18] Note that the lower bound on β in (4) is bounded with respect to δ which means we can ensure $\phi(m) < 1$ as $\delta \to 1$.

[19] In some situations, over-reporting may be deterred by requiring invoices for reported sales or a customer list which could then be contacted. There is no analogous verification process to detect under-reporting because a firm could withhold invoices or deliver a subset of its true customer list. In other words, evidence can be provided to prove that a sale was made but it is not possible to provide evidence that a sale wasn't made. This discussion suggests that the challenge is to provide incentives to discourage under-reporting.

[20] It is established in the proof of Theorem 1 that, when ϕ is weakly convex, if it is not optimal to under-report by one unit, then it is not optimal to under-report by any amount. Also, given (3), (5) is the same regardless of $\left(q, \underline{p}\right)$.

$$\widehat{p} = p^N \left(c + \left(\frac{n}{n-1} \right) z \right) \Leftrightarrow z = \left(\frac{n-1}{n} \right) \left[p^{N^{-1}} \left(\widehat{p} \right) - c \right],$$

in which case (1) is equivalent to

$$\frac{\left[\widehat{p} - p^N \left(c \right) \right] \left(\mu / n \right)}{z} > \overline{m} - \mu. \tag{6}$$

Since

$$V - V^N = \frac{\left[\widehat{p} - p^N \left(c \right) \right] \left(\mu / n \right)}{1 - \delta + \delta \sum_{m=\underline{m}}^{\overline{m}} \rho \left(m \right) \phi \left(m \right)} \tag{7}$$

then the foregone loss from cartel breakdown is increasing in the collusive price $\widehat{p}$, and is decreasing in the probability of cartel breakdown, $\sum_{m=\underline{m}}^{\overline{m}} \rho \left(m \right) \phi \left(m \right)$. As $\widehat{p}$ is increasing in the per unit transfer z, this argues for a higher value of z in order to raise the average gain in profit from maintaining collusion, $\left[\widehat{p} - p^N \left(c \right) \right] \left(\mu / n \right)$. At the same time, z is the savings in the payment that a firm has to make when it under-reports by one unit, in which case higher z also increases the incentive to under-report. What is then needed is that $\left[\widehat{p} - p^N \left(c \right) \right] \left(\mu / n \right)$ is large enough relative to z, which is what (6) requires.[21] A second key factor determining the size of $V - V^N$ is the probability of cartel breakdown, which is the expression:

$$\sum_{m=\underline{m}}^{\overline{m}} \rho \left(m \right) \phi \left(m \right) = \sum_{m=\underline{m}}^{\overline{m}} \rho \left(m \right) \beta \left(\overline{m} - m \right) \left(1 - \delta \right) = \beta \left(\overline{m} - \mu \right) \left(1 - \delta \right). \tag{8}$$

(8) needs to be sufficiently small so that $V - V^N$ is sufficiently large and, therefore, (5) holds. Hence, $\overline{m} - \mu$ must be small enough, as expressed in (6). In sum, sustaining collusion requires that: i) the rise in the collusive price from a per unit transfer is sufficiently large relative to the size of the per unit transfer; and ii) the likelihood of low demand is sufficiently small so that cartel breakdown is not too likely.

Although it is possible that there is another equilibrium that relaxes the demand condition in (1), the general intuition seems robust: Demand uncertainty combined with the need to provide incentives to price high and report truthfully requires a combination of transfers and value destruction. In particular, value destruction has to be more likely when a firm reports no sales, for otherwise a firm could price low and report it didn't sell anything. Some inefficiency is then necessary with any collusive scheme. However, if this inefficiency happens too often, value is destroyed to the point that firms find collusion unprofitable. As a result, if the probability of low market demand states is too high, collusion is not feasible with semi-public collusive schemes that do not involve delay in reporting.[22] Consistent with this intuition, in

[21] When $p^N(c) = c + const$, as in Corollary 2, then $\widehat{p} - p^N \left(c \right) = \left(\frac{n}{n-1} \right) z$ and, therefore, z cancels out in (6).

[22] If firms report sales with delay, that reduces the probability of there being no sales over the time interval between meetings. Pooling periods then reduces the uncertainty in aggregate demand and hence reduces the need for inefficient punishments. This intuition originated with Abreu, Milgrom, and Pearce (1991) and was later employed in Compte (1998) and others to prove Folk Theorems. However, since individual sales are informative about the sales of other players, constructing equilibria with delay in communication for general probability distributions over a firm's sales is difficult.

the next section we provide an example whereby condition (1) is tight for semi-public perfect equilibria without delay.

Finally, let us show that, as the period length becomes arbitrarily small, supracompetitive profits can be sustained using the lysine strategy profile. Re-scaling by $1-\delta$, the normalized supracompetitive payoff is $(1-\delta)\left(V-V^N\right)$. Substituting the minimum value of β from (4), we derive:

$$(1-\delta)\left(V-V^N\right)=(1/n)\left[\mu\left(\widehat{p}-p^N(c)\right)-nz\left(\overline{m}-\mu\right)\right].$$

Thus, $(1-\delta)\left(V-V^N\right)>0$ if and only if

$$\mu\left[\widehat{p}-p^N(c)\right]-nz\left(\overline{m}-\mu\right)>0,$$

which holds by (6).

In sum, we have shown that, when prices and quantities are private information, firms can effectively sustain collusion using the type of practices observed by recent cartels in the markets for citric acid, lysine, and vitamins.

5 Optimal Mechanism for the Two Unit Demand Case

The previous section showed that observed collusive practices are consistent with an equilibrium in which firms truthfully report their sales, make transfers to the other firms based on those sales, and for which cartel breakdown is more likely when aggregate reported sales is lower. We now turn to the objective of characterizing an optimal collusive equilibrium. In particular, are there equilibria with a different structure that can sustain higher payoffs for cartel members? Or, have cartel members discovered the most effective collusive practices?

Taking a mechanism design approach to this question, we focus on a highly simplified duopoly case when market demand can take on values of 0, 1 or 2.[23] Reports are restricted so that $r_i \in \{0,1,2\}$. Let $\rho_m \equiv \rho(m)$ be a shorthand for the probability that demand is $m \in \{0,1,2\}$, $\eta_{i,j}(p_1,p_2)$ is the probability that $q_2=j$ given $q_1=i$ and firms' prices, and $\psi(q_1;m,p_1,p_2)$ is the probability of firm 1 having sales of q_1 given total demand is m and given firms' prices.

To make the analysis tractable, we assume independence of the customers' decisions and a probability function that depends only on the difference in firms' prices. In particular, let the probability that any consumer buys from firm 1 be $\xi(p_2-p_1)$. We assume ξ is a differentiable, increasing function and that $\xi(0)=1/2$. Furthermore, ξ is such that, given the scheme we construct, the FOC of the price-setting problem is sufficient for optimality; a sufficient condition is that $\xi'' \leq 0$.

[23] With some additional notation, we believe results can be extended in a straightforward manner to when there are n firms.

Using ξ and the independence of consumer choices we get:

$$\begin{aligned}
\psi(1;1,p_1,p_2) &= \xi(p_2-p_1)\\
\psi(2;2,p_1,p_2) &= \xi(p_2-p_1)^2\\
\psi(1;2,p_1,p_2) &= 2\xi(p_2-p_1)[1-\xi(p_2-p_1)].\\
\psi(0;2,p_1,p_2) &= [1-\xi(p_2-p_1)]^2
\end{aligned}$$

A collusive mechanism consists of a recommended price pair (p_1,p_2) and a transfer rule that depends on reported sales. A transfer rule $\{t_1(r_1,r_2),\ t_2(r_2,r_1)\}$ specifies net transfers received by the two players conditional on the reports. The mechanism is feasible if

$$t_1(r_1,r_2)+t_2(r_2,r_1)\leq 0, \forall (r_1,r_2). \tag{9}$$

Moreover, we restrict the transfers to be bounded:

$$t_1(r_1,r_2), t_2(r_2,r_1)\in[-x,x],\ \forall (r_1,r_2), \tag{10}$$

where $x>0$. Restriction (10) may be needed for the existence of an optimal mechanism because total demand is inelastic. If bigger inter-firm transfers are more likely to trigger an investigation by the antitrust authorities, cartel members may want to put a bound on those transfers. Thus, the technical assumption in (10) may have an economic rationale as well. The mechanism is incentive compatible if both firms find it optimal to set the recommended prices and report their realized sales truthfully.

A collusive mechanism is symmetric if $t_1(r_1,r_2)=t_2(r_2,r_1)=t(r_1,r_2)$ and $p_1=p_2=\widehat{p}$. Our goal is to describe an optimal symmetric incentive compatible feasible collusive mechanism.

Anticipating that both firms will truthfully report their sales, firm 1's expected payoff at the price stage is:

$$\begin{aligned}
&\rho_0 t(0,0)+\rho_1[\psi(1;1)(p_1-c+t(1,0))+(1-\psi(1;1))t(0,1)] \\
&+\rho_2[\psi(2;2)(2(p_1-c)+t(2,0))+\psi(1;2)(p_1-c+t(1,1))+(\psi(0;2))t(0,2)],
\end{aligned} \tag{11}$$

where we have suppressed the dependence of $\psi(\cdot)$ on firms' prices. The FOC for price (assuming truthful reporting) gives us the symmetric equilibrium price $\widehat{p}$:

$$\begin{aligned}
0 &= \rho_1\left\{\frac{\partial\psi(1;1)}{\partial p_1}[\widehat{p}-c+t(1,0)-t(0,1)]+\frac{1}{2}\right\} \\
&\quad+\rho_2\left\{\frac{\partial\psi(2;2)}{\partial p_1}[2(\widehat{p}-c)+t(2,0)]+2\psi(2;2)\right.\\
&\quad\left.+\frac{\partial\psi(1;2)}{\partial p_1}[\widehat{p}-c+t(1,1)]+\psi(1;2)+\frac{\partial(\psi(0;2))}{\partial p_1}t(0,2)\right\}.
\end{aligned} \tag{12}$$

Using our assumptions on ψ, this can be simplified to

$$\begin{aligned}
0 &= \rho_1\left\{-\xi'(0)[\widehat{p}-c+t(1,0)-t(0,1)]+\frac{1}{2}\right\} \\
&\quad+\rho_2\left\{-\xi'(0)[2(\widehat{p}-c)+t(2,0)-t(0,2)]+1\right\}.
\end{aligned} \tag{13}$$

Solving it, the symmetric equilibrium price is:

$$\widehat{p} = c + \frac{1}{2\xi'(0)} + \frac{\rho_1}{\rho_1 + 2\rho_2}\left[t(0,1) - t(1,0)\right] + \frac{\rho_2}{\rho_1 + 2\rho_2}\left[t(0,2) - t(2,0)\right]. \quad (14)$$

When $t(0,1) - t(1,0)$ is higher, a firm benefits more from being the firm with zero demand when market demand is one. There is then an incentive for a firm to raise price and that is why the equilibrium price is increasing in $t(0,1) - t(1,0)$. A similar logic explains why the equilibrium price is increasing in $t(0,2) - t(2,0)$.

Let us consider the incentive compatibility constraints (ICCs) in the reporting stage. Suppose $q_1 = 2$, in which case firm 1 knows that firm 2 sold zero units. The ICC for truthful reporting is

$$t(2,0) \geq t(1,0), t(0,0). \quad (15)$$

When $q_1 = 1$, the ICCs for truthful reporting are

$$\eta_{1,1}(p_1, \widehat{p})\, t(1,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(1,0) \geq \eta_{1,1}(p_1, \widehat{p})\, t(2,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(2,0) \quad (16)$$

$$\eta_{1,1}(p_1, \widehat{p})\, t(1,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(1,0) \geq \eta_{1,1}(p_1, \widehat{p})\, t(0,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(0,0) \quad (17)$$

$\eta_{1,1}(p_1, \widehat{p})$ is the probability that firm 1 assigns to firm 2 selling one unit, given firm 1 sold one unit and the price pair. By (16), firm 1 prefers to report having sold one unit than reporting two units; and by (17), firm 1 prefers to report having sold one unit than reporting zero units. It is necessary for the mechanism to be incentive compatible that (16) and (17) hold at $p_1 = \widehat{p}$. However, that is not sufficient, since the firm may have a profitable "double-deviation"; that is, deviating with price *and* report. In our construction in the proof of Theorem 3, we use only these necessary conditions and then verify that the firm has no incentive to misreport even if it deviates in price as well. Finally, when $q_1 = 0$, the ICCs are

$$\begin{aligned} & \eta_{0,2}(p_1, \widehat{p})\, t(0,2) + \eta_{0,1}(p_1, \widehat{p})\, t(0,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(0,0) \\ \geq\ & \eta_{0,2}(p_1, \widehat{p})\, t(1,2) + \eta_{0,1}(p_1, \widehat{p})\, t(1,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(1,0) \end{aligned} \quad (18)$$

$$\begin{aligned} & \eta_{0,2}(p_1, \widehat{p})\, t(0,2) + \eta_{0,1}(p_1, \widehat{p})\, t(0,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(0,0) \\ \geq\ & \eta_{0,2}(p_1, \widehat{p})\, t(2,2) + \eta_{0,1}(p_1, \widehat{p})\, t(2,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(2,0) \end{aligned} \quad (19)$$

(again it is necessary that these hold for $p_1 = \widehat{p}$, and sufficient if they hold for all p_1).

Substituting (14) into the expected payoff in (11), the problem is to choose a transfer function $t(\cdot)$ to maximize

$$\begin{aligned}
&\rho_0 t(0,0) + (\rho_1/2)[t(1,0) + t(0,1)] \\
&+ \left(\frac{1-\rho_0-\rho_1}{4}\right)[t(2,0) + 2t(1,1) + t(0,2)] \\
&+ \left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + \left(\frac{\rho_1}{2}\right)[t(0,1) - t(1,0)] + \left(\frac{1-\rho_0-\rho_1}{2}\right)[t(0,2) - t(2,0)]
\end{aligned} \tag{20}$$

subject to the feasibility constraints (9)-(10) and the ICCs (15)-(19).

Theorem 3 *Under the assumptions of Section 5, if the high demand state is most likely ($\rho_2 > \rho_0, \rho_1$), an optimal symmetric mechanism is:*

$$\begin{aligned}
t(0,0) &= -x \\
t(0,1) &= 0, t(1,0) = -x \\
t(0,2) &= x, t(2,0) = -x \\
t(1,1) &= 0 \\
t(r_1, r_2) &= -x \text{ if } r_1 + r_2 > 2
\end{aligned}$$

and the resulting expected firm payoff is:

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + (\rho_2 - \rho_0)x.$$

If the low demand state is most likely ($\rho_0 > \rho_1, \rho_2$), there does not exist any symmetric mechanism yielding payoffs in excess of those produced by a stage game Nash equilibrium.

When the low demand state is most likely, collusion cannot be sustained.[24] We do not have a characterization when the medium demand state is most likely ($\rho_1 > \rho_0, \rho_2$).[25] When the high demand state is most likely, collusion can be sustained and the optimal mechanism has the following properties. When market demand is two units and one firm sold both of those units, that firm is required to make a transfer of x to the firm that sold nothing. When both firms sold one unit, there are no transfers. When market demand is one unit, the firm having sold that unit incurs a penalty of x and the other firm receives no payment, so value is destroyed. Finally, when market demand is zero, both firms incur a penalty of x, and again there is an inefficiency. The remainder of this section will explore this optimal mechanism; thus, we will be assuming $\rho_2 > \rho_0, \rho_1$.

[24] As earlier work on private monitoring suggests, delay in exchanging reports will presumably be necessary to support collusion when $\rho_0 > \rho_1, \rho_2$.

[25] When $\rho_1 > \rho_0, \rho_2$, we can characterize an optimal mechanism when a firm deviates in its price *or* in its reports, but a mechanism immune to deviating simultaneously in price and report has thus far alluded us. The difficulty is in verifying that there are no profitable double deviations.

Let us first show how this optimal mechanism can be implemented as a semi-public perfect equilibrium of an infinitely repeated game. Define

$$v \equiv \left(\frac{2\left(1-\rho_0\right)-\rho_1}{4\xi'\left(0\right)} \right) + \left(\rho_2 - \rho_0\right)x,\ v^N \equiv \left(\frac{2\left(1-\rho_0\right)-\rho_1}{4\xi'\left(0\right)} \right)$$

as the per period expected payoff for the optimal mechanism and the stage Nash equilibrium, respectively. When both firms report zero sales, each firm is supposed to incur a penalty of x. As the foregone value from going to the stage game Nash equilibrium is $\left(\frac{\delta}{1-\delta}\right)\left(v-v^N\right)$, we then want to realize that penalty with a probability such that the expected foregone value equals $t\left(0,0\right)$. Hence, when $\left(r_1, r_2\right) = \left(0, 0\right)$, the equilibrium shifts to the stage game Nash equilibrium forever with probability α_0 which satisfies:

$$x = \alpha_0 \left(\frac{\delta}{1-\delta}\right)\left(v-v^N\right) \Leftrightarrow \alpha_0 = \frac{1-\delta}{\delta\left(\rho_2-\rho_0\right)}.$$

If $\left(r_1, r_2\right) = \left(1, 0\right)$ then firm 1 is to pay x and firm 2 has a zero transfer. To implement it, assume firm 1 transfers $x/2$ to firm 2 and the probability the equilibrium shifts to stage game Nash forever is α_1 which satisfies:

$$\frac{x}{2} = \alpha_1 \left(\frac{\delta}{1-\delta}\right)\left(v-v^N\right) \Leftrightarrow \alpha_1 = \frac{1-\delta}{2\delta\left(\rho_2-\rho_0\right)}.$$

Thus, firm 1 incurs a penalty of x - as it pays $x/2$ to firm 2 and incurs an expected loss of $x/2$ from possible cartel breakdown - while firm 2 experiences no net transfer as it receives $x/2$ from firm 2 but incurs an expected loss of $x/2$ from possible cartel breakdown. Finally, if $\left(r_1, r_2\right) = \left(2, 0\right)$ then firm 1 simply transfers x to firm 2. This strategy profile implements the optimal mechanism and is an equilibrium iff

$$\frac{1-\delta}{\delta\left(\rho_2-\rho_0\right)} \leq 1 \Leftrightarrow \delta \geq \frac{1}{1+\rho_2-\rho_0}.$$

Note that the smaller is $\rho_2 - \rho_0$, the more patient firms have to be.

To summarize, assume $\rho_2 > \rho_0, \rho_1$ and firms are sufficiently patient,

$$\delta \geq \frac{1}{1+\rho_2-\rho_0}.$$

Substituting the transfer function from Theorem 3 into (14), the equilibrium price is

$$\widehat{p} = c + \frac{1}{2\xi'\left(0\right)} + x.$$

The equilibrium probability of cartel breakdown is

$$\phi\left(r_1+r_2\right) = \begin{cases} \frac{1-\delta}{\delta\left(\rho_2-\rho_0\right)} & \text{if } r_1+r_2=0 \\ \frac{1-\delta}{2\delta\left(\rho_2-\rho_0\right)} & \text{if } r_1+r_2=1 \\ 0 & \text{if } r_1+r_2=2 \\ \frac{1-\delta}{\delta\left(\rho_2-\rho_0\right)} & \text{if } r_1+r_2>2 \end{cases}$$

and inter-firm payments are:

r_1	r_2	Payment from firm 1 to firm 2
0	0	0
1	0	$x/2$
1	1	0
2	0	x

The properties of this optimal equilibrium match those of the lysine strategy profile quite closely. First, the payment scheme is linear in the number of units; a firm transfers an amount $x/2$ to the other cartel member for each unit it reports having sold. Of particular note is that payments depend only on a firm's own sales report. Second, the probability of cartel breakdown depends only on the aggregate sales report, and is linear for equilibrium values:

$$\phi(r_1 + r_2) = \left[\frac{1-\delta}{2\delta(\rho_2 - \rho_0)}\right](2 - r_1 - r_2).$$

The similarity between the optimal equilibrium and observed collusive practices suggests that cartels have identified highly effective methods for colluding when prices and quantities are private information. The ubiquity of these practices may well be due to the fact that there may not be practices which are more efficacious.

6 Concluding Remarks

As it has been understood for a very long time, monitoring of a collusive agreement is essential to its success. Only if firms can expect non-compliance to be observed and punished will cartel members abide by the agreement to maintain high prices and limit supply. In spite of the critical role of monitoring, there are many well-documented episodes of successful collusion in markets for which there is limited public information to use for the purposes of monitoring compliance. In environments for which prices and sales are private information, it was frequently observed that collusion involved a similar set of practices: firms self-report sales and conduct inter-firm transfers based on those reports; specifically, firms with high sales compensate those firms with low sales. The main contribution of this paper is showing how those practices can result in successful collusion in this informationally-scarce environment. To induce firms to set a collusive price, the equilibrium strategy has a firm make a payment to the other firms for each unit that it reports having sold. To induce truthful reporting of sales, the probability of cartel breakdown - reversion to a stage game Nash equilibrium - is specified to be higher when total reported market sales is lower. Thus, a firm that under-reports its sales realizes a benefit by having to make a lower payment to the other firms, but also incurs a cost by increasing the probability that collusion breaks down. In equilibrium, the cost of the increased likelihood of cartel breakdown exceeds the benefit from reduced inter-firm payments, so that a firm finds it optimal to truthfully report its sales. In the special case when market demand is zero, one or two units, we characterized an optimal mechanism which has much the same properties of this equilibrium. Observed collusive practices are then not only consistent with equilibrium but also with an optimal mechanism for colluding. That various cartels have responded in a similar manner to a problematic monitoring environment may be due to them having identified optimal practices.

When the market involves intermediate goods - so that customers are industrial buyers and thus price is private information between a buyer and a seller - our theory suggests what antitrust authorities should look for in terms of collusive practices and outcomes. First, firms exchanging sales reports. While this can occur in secret, it can also occur through a trade association. Second, inter-firm sales or other forms of compensation, as the use of asymmetric punishments is essential to effective collusion (a point originally made in Harrington and Skrzypacz, 2007). Third, periodic price wars as it is the possible threat of a price war that induces firms to truthfully report their sales. Thus, information exchange, inter-firm transfers, and periodic price wars all add up to collusion.

7 Appendix A: Folk Theorems with Private Monitoring and Communication

There is a growing literature on Folk Theorems in a repeated game when the history of actions is not common knowledge. This literature can be partitioned into work that allows players to communicate through cheap talk messages and work that does not (an example of the latter is Hörner and Olszewski, 2006). As cartels did engage in costless communication - and our equilibria will allow for it - we will limit our attention to reviewing research on private monitoring with communication.[26]

Consider a repeated game in which, in each period, a player receives a private signal of the actions selected by the other players in the previous period and then players simultaneously choose actions. A player's private history comprises her past signals and actions; there is no public history. A player's payoff depends only on her action and signal so, once observing the signal, the payoff contains no information about other players' actions. Now, augment this structure by allowing players to send costless messages. After observing their private signals and prior to choosing an action, players simultaneously send messages that are publicly observed. Based on these messages, monetary payments are allowed between players and, in some cases, "burning money" is permitted (that is, net transfers are negative).

In developing a Folk Theorem, there are three primary issues that need to be addressed: 1) Are the private signals sufficient to statistically detect cheating by a player?; 2) How are players induced to truthfully reveal their private signals?; and 3) If punishments occur in equilibrium with positive probability, how is efficiency achieved? In our brief survey, we will touch on how these issues are tackled by the literature.

In a setting with three or more players, Kandori and Matsushima (1998) address the first issue by assuming that, for any pair of players, the remaining players' private signals can statistically determine which of those two players cheated. To deal with the second issue, they use the trick of making a player's net payment depend only on the messages of the other players. Since a player's net payment is independent of her own message, she has a (weak) incentive to truthfully reveal her private signal. Finally, efficiency is achieved as the scheme has players make pair-wise transfers so there is no burning money in equilibrium (and no other form of punishment). All this delivers a Folk Theorem. From our perspective of explaining cartel practices, there are three weaknesses to this scheme. First, that a firm's payment does not depend on its own reported sales is inconsistent with documented collusive practices. Second, the statistical detection assumption is strong and, in the oligopoly context, is unlikely to be satisfied. For example, if firm 1 cheats by pricing low in a price game, firm 3 may be able to statistically detect that someone cheated - because firm 3's quantity is low - but will be unable to determine whether it was firm 1 or 2.[27] Third, the

[26] For earlier reviews of this literature, see Kandori (2002) and Mailath and Samuelson (2006).

[27] They correctly note that if firms' products are asymmetrically differentiated then their assumption will be satisifed. But that is of value to explaining actual cartel practices only if the degree of asymmetric differentiation is not trivial. As most price-fixing cartels involve homogeneous commodities - such as chemicals - their assumption is not particularly palatable.

mechanism requires at least three players.[28]

Compte (1998) assumes that a player can statistically detect cheating but does not require statistical determination of who cheated; thus doing away with the strong assumption of Kandori and Matsushima (1998). However, this is at the cost of requiring that players' signals are independent. Again, players make payments based on reports though these transfers do not necessarily balance which means that money is burned. This inefficiency is eliminated, however, by having players exchange informative messages with delay. As information is exchanged and money is burnt in the distant future, the inefficiency disappears as the discount factor goes to one (and the delay goes to infinity). The assumption that players' signals are independent limits the usefulness of his scheme for modelling cartel behavior. In a price-setting oligopoly game, a firm's signal is its sales and one would expect firms' sales to be positively correlated due to common demand shocks.

Aoyagi (2002) considers a Bertrand price game where firms' demands are affiliated and their private signals are correlated. To get a Folk Theorem, he assumes firm demand is discontinuous in price at the equilibrium price vector, while we assume that demand is continuous in prices. His assumptions may be appropriate for cartels operating in auction markets, but would not seem right for product markets such as chemicals which is the motivation for our analysis. Furthermore, the structure of equilibrium is rather unnatural. The equilibrium partitions the horizon into T components - where component k is made up of periods k, $T+k$, $2T+k, \ldots$ - and a firm's behavior is assumed to influence future behavior only within a component. For example, if a firm prices low in period k and this produces low sales for the other firms, the induced punishment is confined to periods $T+k$, $2T+k, \ldots$. There is no evidence this type of mechanism corresponds with actual collusive practices.

The recent work of Zheng (2008) and Obara (2009) have the most general assumptions on players' signals. Both studies are similar in their structure to that of Compte (1998) but dispense with the requirement that players' signals are independent. When there is delay - that is, multiple private signals are received before messages are sent - there is a tricky learning issue which must be circumvented. As a player accumulates signals, she may update her beliefs about the signals received by other players which will affect her beliefs over the likelihood of a punishment. Thus, the incentive to deviate can evolve and this can make it difficult to construct an equilibrium. Compte (1998) rules out such learning by assuming players' signals are independent. Zheng (2008) replaces independence with what he refers to as "effective independence" in that signals are allowed to be correlated but the equilibrium is constructed so that the incentive compatibility constraint is unchanged as signals are received. He adjusts the punishment by having the probability of going to the stage game Nash equilibrium to depend in a precise way on reported messages. A problem in applying Zheng (2008) to cartels is that he has a correlation condition which can easily be violated in the oligopoly setting. For example, it does not hold when market demand is not very volatile so that there is a low probability of all firms

[28] They also provide a different scheme for when there are two players which is related to the work of Compte (1998) which we review next.

having a low level of sales.[29] Also, the assumption that the vector of signals lies in a product space, which is important for the equilibrium construction, is violated when market demand is stochastically realized and then allocated among firms, as then the sum of firms' signals must be less than maximal market demand.

Obara (2009) also dispenses with the restriction that players' signals are independent. In its place, it is assumed that, for each player, there is at least one player that can statistically detect a deviation by that player. Though this condition may be plausible for some generic games, it may be hard to satisfy for a symmetric oligopoly game. Furthermore, an unattractive feature to the equilibrium characterized is its asymmetric structure in that different players perform different roles. For example, in the two-player setting, player 1 provides a payment to player 2 which is based only on player 1's signal; thus, player 2 has a weak incentive to provide an informative message. However, player 1 receives a payment based on both players' signals. To induce him to be truthful, there is a probabilistic punishment based on player 1's message. This ex ante asymmetric treatment of players does not fit well with what we know about cartel behavior.

8 Appendix B: Static Nash Equilibrium

Assume that $\psi_i\left(q;m,\underline{p}\right)$ depends only on pair-wise price differences; this property can be derived from a consumer choice model with quasi-linear preferences in money. It implies that for all prices:

$$\psi_i\left(q;m,\underline{p}\right)=\psi_i\left(q;m,\underline{p}+\Delta 1_n\right) \tag{21}$$

where 1_n is a vector of ones of length n. Also, assume that the FOC is sufficient to characterize a unique equilibrium for any symmetric cost structure. Let $\psi_i\left(m,\underline{p}\right)$ denote the corresponding vector specifying the probabilities over different quantities $q_i \in \{0,...\overline{m}\}$ for player i.

Lemma 4 *The equilibrium path-through of an increase in marginal costs to prices is 100%. That is, if all marginal costs increase by Δ, then the Nash equilibrium prices increase by Δ.*

Proof. Firm i's best response problem to set price $p_i^N(c_i)$ to other firms setting prices p_{-i}^N is:

$$p_i^N(c_i) \in \arg\max_{p_i} \sum_{m=\underline{m}}^{\overline{m}} \rho(m)(p_i-c_i)\left(\underline{q_i}\cdot\psi_i\left(m,p_1^N,\ldots,p_i,\ldots,p_m^N\right)\right).$$

[29] Examining equation (6) from Zheng (2008), suppose y' is a low level of sales, y is a modest level of sales, and a_i is a price below a^*. Assume the probability that all firms have sales of y' is zero (or close to zero); then the rhs expression is unbounded. In contrast, the lhs expression is bounded when the probability that other firms have low sales - conditional on the low priced firm having modest sales - is bounded above zero.

where $\underline{q_i}$ is a vector $(0, 1, ...\overline{m})$ and "$\cdot$" is a dot product. The FOC of this problem is

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\left(\underline{q_i} \cdot \psi_i \left(m, \underline{p^N}\right) + (p_i - c_i) \underline{q_i} \cdot \frac{\partial \psi_i \left(m, \underline{p^N}\right)}{\partial p_i} \right) \right] = 0$$

Suppose it is satisfied at $\underline{p^N}(c)$. Consider costs $c + \Delta$ and evaluate the FOC at prices $\underline{p^N}(c) + \Delta 1_n$:

$$\begin{aligned} &\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\left(\underline{q_i} \cdot \psi_i \left(m, \underline{p^N} + \Delta 1_n\right) + ((p_i + \Delta) - (c_i + \Delta)) \underline{q_i} \cdot \frac{\partial \psi_i \left(m, \underline{p^N} + \Delta 1_n\right)}{\partial p_i} \right) \right] \\ = &\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\left(\underline{q_i} \cdot \psi_i \left(m, \underline{p^N}\right) + (p_i - c_i) \underline{q_i} \cdot \frac{\partial \psi_i \left(m, \underline{p^N}\right)}{\partial p_i} \right) \right] = 0 \end{aligned}$$

where we used (21) to simplify. Hence, the FOC holds at these prices. ■

100% pass-through implies that $p^N(c) = c + const$. Finally, note that the above lemma holds even if costs are asymmetric: when firms have a vector of marginal costs $\underline{c}$ then $\underline{p}^N(\underline{c} + \Delta 1_n) = \underline{p}^N(\underline{c}) + \Delta 1_n$.

9 Appendix C: Proofs

9.1 Proof of Theorem 1

In proving that the lysine strategy profile is a semi-public perfect equilibrium, we need to establish: i) given any history of reports and payments, the prescribed price is optimal; ii) given any history of reports and payments and, for the current period, a firm's price and sales, the prescribed report is optimal; and iii) given any history of reports and payments (including the sales reports of firms in the current period), the prescribed payment is optimal. We will tackle them in that sequence with the bulk of the analysis concerning the incentive compatibility of sales reports, which is step (ii). When the public history has one or more firms not submitting reports or one or more firms not making payments, the optimality of behavior is obvious because the industry is to be in the non-collusive phase. Our attention will then focus on public histories on the equilibrium path, which means firms are in the collusive phase and all firms submitted reports and made the appropriate payments.

By symmetry, let us restrict the analysis to firm 1. Given a generic price vector $\underline{p}$ and the anticipation that truthful reports will be submitted and transfers will be made, firm 1's payoff at the price stage is

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q; m, \underline{p}) \left\{ \left[(p_1 - c) q + z \left(\frac{m-q}{n-1} \right) - zq \right] + \phi(m) \delta V^N + (1 - \phi(m)) \delta V \right\}. \tag{22}$$

V is the value when in the collusive phase and V^N is the value when in the non-collusive phase. Note that a firm with sales q expects to pay zq, while receiving an equal share of the payments made by the other $n-1$ firms which equal $z(m-q)$. Simplifying this expression, we derive

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q;m,\underline{p}) \left(p_1 - c - \left(\frac{n}{n-1}\right) z\right) q$$
$$+ \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \left[\phi(m)\delta V^N + (1-\phi(m))\delta V\right] + z\left(\frac{\mu}{n-1}\right)$$

Thus, if the collusive price is $\widehat{p}$ then the incentive compatibility constraint (ICC) at the price stage is

$$\sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q;m,\underline{\widehat{p}}) \left(\widehat{p} - c - \left(\frac{n}{n-1}\right) z\right) q \tag{23}$$
$$\geq \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \sum_{q=0}^{m} \psi_1(q;m,p_1,\widehat{p},\ldots,\widehat{p}) \left(p_1 - c - \left(\frac{n}{n-1}\right) z\right) q, \ \forall p_1$$

(23) is satisfied iff $\widehat{p} = p^N\left(c + \left(\frac{n}{n-1}\right)z\right)$; that is, $\widehat{p}$ is the static Nash equilibrium when unit cost is $c+\left(\frac{n}{n-1}\right)z$. For any z, we will then assume $\widehat{p} = p^N\left(c+\left(\frac{n}{n-1}\right)z\right)$. Since $p^N\left(c+\left(\frac{n}{n-1}\right)z\right)$ is continuously increasing and unbounded in z by A4, then for any $p > p^N$ there exists $z > 0$ such that $p = p^N\left(c+\left(\frac{n}{n-1}\right)z\right)$.

Taking account of the fact that transfers average out to zero, expected collusive profit is

$$\widehat{\pi}(z) \equiv \left[p^N\left(c + \left(\frac{n}{n-1}\right)z\right) - c\right]\left(\frac{\mu}{n}\right).$$

'The collusive value is recursively defined by:

$$V = \widehat{\pi}(z) + \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\left[\phi(m)\delta V^N + (1-\phi(m))\delta V\right].$$

The incremental gain from being in the collusive phase is

$$V - V^N = \widehat{\pi}(z) + \sum_{m=\underline{m}}^{\overline{m}} \rho(m)\left[\phi(m)\delta V^N + (1-\phi(m))\delta V\right] - \widehat{\pi}(0) - \delta V^N.$$

Solving for $V - V^N$,

$$V - V^N = \frac{\widehat{\pi}(z) - \widehat{\pi}(0)}{1 - \delta + \delta\sum_{m=\underline{m}}^{\overline{m}} \rho(m)\phi(m)} = \frac{\left[p^N\left(c+\left(\frac{n}{n-1}\right)z\right) - p^N(c)\right](\mu/n)}{1-\delta+\delta\sum_{m=\underline{m}}^{\overline{m}}\rho(m)\phi(m)}. \tag{24}$$

This expression will be useful later in the proof.

The next step is to consider the optimality of a firm's strategy at the report stage. If in the collusive phase then firm 1's beliefs put unit mass on other firms setting a price equal to $\widehat{p}$. Firm 1's strategy is sequentially rational if the prescribed report is optimal given the other firms' current period price was $\widehat{p}$, any arbitrary price for firm 1, and any feasible level of realized sales for firm 1, which is denoted q_1.[30]

Recall that $\sigma_1 \left(m \left| q_1, \underline{p}\right.\right)$ denotes firm 1's posterior beliefs on total demand conditional on its quantity and the price vector. Given the other firms are expected to provide a truthful report and make payments, firm 1's expected payoff from reporting r_1 (when its true sales is q_1) is[31]

$$\begin{aligned} W\left(r_1; q_1, p_1\right) \equiv & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| q_1, \underline{p}\right.\right) \left\{ \left[\left(p_1 - c\right) q_1 + z \left(\left(\frac{m - q_1}{n-1} \right) - r_1 \right) \right] \right. \\ & \left. + \phi\left(m - q_1 + r_1\right) \delta V^N + \left(1 - \phi\left(m - q_1 + r_1\right)\right) \delta V \right\} \\ = & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| q_1, \underline{p}\right.\right) \left\{ \left[\left(p_1 - c\right) q_1 + \left(\frac{z}{n-1} \right) \left(m - q_1 - (n-1) r_1\right) \right] \right. \\ & \left. + \phi\left(m - q_1 + r_1\right) \delta V^N + \left(1 - \phi\left(m - q_1 + r_1\right)\right) \delta V \right\}. \end{aligned}$$

The ICCs are:

$$W\left(q_1; q_1, p_1\right) \geq W\left(r_1; q_{1,}, p_1\right) \ \forall r_1 \in \{0, 1, \ldots\}, \ \forall q_1 \in \{0, 1, \ldots, \overline{q}\}, \ \forall p_1. \tag{25}$$

These ICCs ensure that, for any price set by firm 1 and any realized sales for firm 1, firm 1 finds it optimal to truthfully report its sales. That statement presumes that firm 1 will make a payment of zr_1 regardless of what r_1 is, which we will later show to be optimal.

Our analysis proceeds through several steps. First, we derive a condition ensuring that it is not optimal to over-report sales; that is, $r_1 = q_1$ is preferred to any $r_1 > q_1$. In deriving a condition ensuring that it is not optimal to under-report - that is, $r_1 = q_1$ is preferred to any $r_1 < q_1$ - we first derive a condition whereby if it is not optimal to under-report by one unit then it is not optimal to under-report by any amount. Then we derive a condition whereby if it is not optimal to under-report by one unit when $q_1 = \overline{q}$ (so a firm's sales are at its maximum level) then it is not optimal to under-report by one unit when $q_1 < \overline{q}$. We are then left with the property: if

$$W\left(\overline{q}; \overline{q}, p_1\right) \geq W\left(\overline{q} - 1; \overline{q}, p_1\right), \ \forall p_1, \tag{26}$$

then all under-reporting ICCs at the report stage are satisfied. Theorem 1 is derived by examining (26) as $\delta \to 1$. As we'll see, its satisfaction requires imposing certain properties on ϕ.

[30] These conditions are sufficient but not necessary for semi-perfect public equilibrium. It is possible there is a deviant price after which reporting truthfully is not a best response, but that deviant price (along with the optimal report) is less profitable than setting the collusive price and reporting truthfully.

[31] Though semi-public perfect equilibrium only requires that the ICC holds when other firms are believed to price at $\widehat{p}$, we will allow for any price vector in order to reduce the amount of notation.

Let us start by deriving a condition which ensures that firm 1 prefers to provide a truthful sales report to over-reporting sales. The ICC is

$$\begin{aligned} & \sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|q_1,\underline{p}\right.\right)\left\{\left[\left(p_1-c\right)q_1+\left(\frac{z}{n-1}\right)\left(m-nq_1\right)\right]\right. \\ & \left.+\phi\left(m\right)\delta V^N+\left(1-\phi\left(m\right)\right)\delta V\right\} \\ \geq\ & \sum_{m=\underline{m}}^{\overline{m}} \sigma\left(m\left|q_1,\underline{p}\right.\right)\left\{\left[\left(p_1-c\right)q_1+\left(\frac{z}{n-1}\right)\left(m-q_1-\left(n-1\right)r_1\right)\right]\right. \\ & \left.+\phi\left(m-q_1+r_1\right)\delta V^N+\left(1-\phi\left(m-q_1+r_1\right)\right)\delta V\right\},\ \forall r_1>q_1. \end{aligned} \tag{27}$$

Performing some manipulations,

$$\begin{aligned} & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\times \\ & \left[\phi\left(m\right)\delta V^N+\left(1-\phi\left(m\right)\right)\delta V-\phi\left(m-q_1+r_1\right)\delta V^N-\left(1-\phi\left(m-q_1+r_1\right)\right)\delta V\right] \\ \geq\ & \sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left(\frac{z}{n-1}\right)\left(m-q_1-\left(n-1\right)r_1-m+nq_1\right) \end{aligned}$$

$$\delta\left(V-V^N\right)\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi\left(m\right)\right]\geq\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)z\left(q_1-r_1\right)$$

$$\delta\left(V-V^N\right)\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi\left(m\right)\right]\geq z\left(q_1-r_1\right). \tag{28}$$

Interpreting (28), the rhs is the change in transfer from reporting r_1 instead of q_1. As $z>0$ and $r_1>q_1$, it is negative. The lhs is the expected change in the future payoff due to over-reporting. It captures the impact of an over-report on the probability of transiting to the non-collusive phase. Equilibrium requires that the change in the expected future payoff is at least as great as the reduction in the current payoff from making a higher payment. As the rhs of (28) is negative, a sufficient condition for (28) to hold is then:

$$\delta\left(V-V^N\right)\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\phi\left(m-q_1+r_1\right)-\phi\left(m\right)\right]\geq 0. \tag{29}$$

We will return to this condition later in the proof.

Next, consider the case of under-reporting. The ICCs are

$$W\left(q_1;q_1,p_1\right)\geq W\left(r_1;q_1,p_1\right)\ \ \forall r_1\in\left\{0,1,\ldots,q_1-1\right\},\ \forall q_1\in\left\{0,1,\ldots,\overline{q}\right\},\ \forall p_1. \tag{30}$$

Given q_1, reporting r_1 is preferred to $r_1 - 1$, where $r_1 \leq q_1$, iff

$$\begin{aligned}
&\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left\{\left[\left(p_1-c\right) q_1+\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1) r_1\right)\right]\right. \\
&\left.+\phi\left(m-q_1+r_1\right) \delta V^N+\left(1-\phi\left(m-q_1+r_1\right)\right) \delta V\right\} \\
\geq &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left\{\left[\left(p_1-c\right) q_1+\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1)\left(r_1-1\right)\right)\right]\right. \\
&\left.+\phi\left(m-q_1+r_1-1\right) \delta V^N+\left(1-\phi\left(m-q_1+r_1-1\right)\right) \delta V\right\} .
\end{aligned} \tag{31}$$

Performing some manipulations,

$$\begin{aligned}
&\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1) r_1\right)-\phi\left(m-q_1+r_1\right) \delta\left(V-V^N\right)\right] \\
\geq &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\left(\frac{z}{n-1}\right)\left(m-q_1-(n-1)\left(r_1-1\right)\right)-\phi\left(m-q_1+r_1-1\right) \delta\left(V-V^N\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)\right] \delta\left(V-V^N\right) \\
\geq &\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left(\frac{z}{n-1}\right)\left[m-q_1-(n-1)\left(r_1-1\right)-m+q_1+(n-1) r_1\right]
\end{aligned}$$

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q_1, \underline{p}\right.\right)\left[\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)\right] \delta\left(V-V^N\right) \geq z . \tag{32}$$

By under-reporting by one unit, firm 1 reduces its payment by z, which is the rhs of (32). The lhs is the expected change in the future payoff from under-reporting.

What we want to show is that if (32) holds for $r_1 = q_1$ then it holds $\forall r_1 < q_1$, and this is true $\forall q_1 \leq \overline{q}$. This is indeed the case if

$$\phi\left(m-q_1+r_1-1\right)-\phi\left(m-q_1+r_1\right)$$

is non-increasing in r_1 $\forall r_1 \leq q_1$, $\forall q_1 \leq \overline{q}$; or, equivalently, $\phi(m-1) - \phi(m)$ is non-increasing in m $\forall m \leq \overline{m}$. From hereon, this property is assumed for ϕ.

Having derived a sufficient condition on ϕ whereby if a firm doesn't want to under-report by one unit then it doesn't want to under-report by any amount, the next step is to derive a sufficient condition such that if it is not optimal to under-report by one unit given $q_1 = q'$, then it is not optimal to under-report by one unit given $q_1 = q' - 1$. Using the explicit expressions, $W(q'; q', p_1) \geq W(q'-1; q', p_1)$ takes the form:

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1\left(m\left|q', \underline{p}\right.\right)[\phi(m-1)-\phi(m)] \geq \frac{z}{\delta\left(V-V^N\right)} . \tag{33}$$

What we want to show is that if (33) holds then (34) holds,

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| q' - 1, \underline{p}\right.\right) [\phi(m-1) - \phi(m)] \geq \frac{z}{\delta (V - V^N)}. \tag{34}$$

This is the case iff the lhs of (34) is at least as great as the lhs of (33):

$$\sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| q' - 1, \underline{p}\right.\right) [\phi(m-1) - \phi(m)] \geq \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| q', \underline{p}\right.\right) [\phi(m-1) - \phi(m)]. \tag{35}$$

Since we've already assumed $\phi(m-1) - \phi(m)$ is non-increasing in m, (35) holds if $\sigma\left(\cdot \left| q', \underline{p}\right.\right)$ FOSD $\sigma\left(\cdot \left| q' - 1, \underline{p}\right.\right)$, which is true by A3.

To summarize, if $\phi(m-1) - \phi(m)$ is non-increasing in m $\forall m \leq \overline{m}$ then (30) holds iff $W(\overline{q}; \overline{q}, p_1) \geq W(\overline{q} - 1; \overline{q}, p_1)$ or

$$\delta \left(V - V^N\right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| \overline{q}, \underline{p}\right.\right) [\phi(m-1) - \phi(m)] \geq z. \tag{36}$$

Substituting (24) into (36), we have

$$\delta \left(\frac{\left[\widehat{p} - p^N(c)\right](\mu/n)}{1 - \delta + \delta \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \phi(m)} \right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| \overline{q}, \underline{p}\right.\right) [\phi(m-1) - \phi(m)] \geq z \tag{37}$$

Summarizing the proof of Theorem 1 thus far, if the ICCs for the payment stage are satisfied (which is shown later) then the lysine strategy profile is a semi-public perfect equilibrium when: i) $\widehat{p} = p^N\left(c + \left(\frac{n}{n-1}\right) z\right)$; ii) $\phi(m-1) - \phi(m)$ is non-increasing in m $\forall m \leq \overline{m}$; iii) (29) holds; and iv) (37) holds. We now want to impose properties on ϕ so that (ii)-(iv) are satisfied as $\delta \to 1$.

Assume

$$\phi(m) = \begin{cases} \beta(\overline{m} - m)(1 - \delta) & \text{if } m \leq \overline{m} \\ (1 - \delta)^{\omega} & \text{if } \overline{m} < m \end{cases} \tag{38}$$

where $\beta > 0$ and $\omega \in (0, 1)$; see Figure 1. With this specification, ϕ is decreasing in m for $m \leq \overline{m}$, equals zero for $m = \overline{m}$, and is positive and constant for $m > \overline{m}$. Note that (ii) is then satisfied. Since $\lim_{\delta \to 1} \phi(m) = 0$ for $m \leq \overline{m}$ then the equilibrium probability of a punishment goes to zero. As we will suppose $\delta \to 1$, $\phi(m)$ is assured of lying in $[0, 1)$, as long as β is bounded.

Inserting (38) into (37), we get

$$\delta \left(\frac{\left[\widehat{p} - p^N(c)\right](\mu/n)}{1 - \delta + \delta \sum_{m=\underline{m}}^{\overline{m}} \rho(m) \beta(\overline{m} - m)(1 - \delta)} \right) \sum_{m=\underline{m}}^{\overline{m}} \sigma_1 \left(m \left| \overline{q}, \underline{p}\right.\right) \beta(1 - \delta) \geq z.$$

Simplifying and re-arranging it, this ICC holds iff

$$\beta \left[\mu\left(\widehat{p} - p^N(c)\right) - nz(\overline{m} - \mu)\right] \geq \frac{nz}{\delta} \tag{39}$$

If

$$\mu\left(\widehat{p}-p^{N}(c)\right)>nz\left(\overline{m}-\mu\right) \tag{40}$$

then (39) is satisfied if β is sufficiently large. Let us show that (40) is equivalent to (1). Given $\widehat{p}=p^{N}\left(c+\left(\frac{n}{n-1}\right)z\right)$, we can solve for the per unit transfer z required to induce the collusive price $\widehat{p}$.

$$p^{N}\left(c+\left(\frac{n}{n-1}\right)z\right)=\widehat{p}\Leftrightarrow c+\left(\frac{n}{n-1}\right)z=p^{N^{-1}}\left(\widehat{p}\right)\Leftrightarrow$$

$$z=\left(\frac{n-1}{n}\right)\left[p^{N^{-1}}\left(\widehat{p}\right)-c\right]. \tag{41}$$

$p^{N^{-1}}$ exists by A4. Substituting (41) into (40) and re-arranging gives us (1). In sum, if (1) holds and ϕ satisfies (38)-(39) then all of the under-reporting ICCs hold.

This leaves us just having to show (iii). This we will do by showing that (29) holds as $\delta\rightarrow 1$. Using (38) and assuming $r_1>q_1$, the lhs of (29) is

$$\delta\left(V-V^{N}\right)\left\{\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[\beta\left(\overline{m}-(m-q_1+r_1)\right)(1-\delta)-\beta\left(\overline{m}-m\right)(1-\delta)\right]\right.$$
$$\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[(1-\delta)^{\omega}-\beta\left(\overline{m}-m\right)(1-\delta)\right]\right\}$$

$$=\delta\left(V-V^{N}\right)\left\{\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)(q_1-r_1)\beta(1-\delta)\right.$$
$$\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[(1-\delta)^{\omega}-\beta\left(\overline{m}-m\right)(1-\delta)\right]\right\}. \tag{42}$$

Using (24) and (38), we substitute for $V-V^{N}$ in (42),

$$=\delta\left(\frac{\left[\widehat{p}-p^{N}(c)\right](\mu/n)}{1-\delta+\delta\sum_{m=\underline{m}}^{\overline{m}}\rho(m)\beta\left(\overline{m}-m\right)(1-\delta)}\right)\times \tag{43}$$
$$\left\{\sum_{m=\underline{m}}^{\overline{m}-(r_1-q_1)}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)(q_1-r_1)\beta\right.$$
$$\left.+\sum_{m=\overline{m}-(r_1-q_1)+1}^{\overline{m}}\sigma_1\left(m\left|q_1,\underline{p}\right.\right)\left[(1-\delta)^{\omega-1}-\beta\left(\overline{m}-m\right)\right]\right\}.$$

Letting $\delta\rightarrow 1$, the first term in $\{\cdot\}$ is bounded, while the second term is unbounded and positive since $\omega-1<0$ implies $\lim_{\delta\rightarrow 1}(1-\delta)^{\omega-1}=+\infty$. (It is here where we use A2.) Hence, as $\delta \to 1$ then the expression in $\{\cdot\}$ is positive. Thus, (iii) holds as $\delta \to 1$.

The final step in proving the lysine strategy profile is a semi-public perfect equilibrium is to show that the prescribed behavior for the payment stage is optimal. First note that reporting zero and following the equilibrium strategy (which entails a zero payment) is at least as good as any positive sales report and not making the corresponding payment; for the latter results in a punishment for sure while the former could have a punishment with probability less than one. Since we've already shown that reporting truthfully is weakly preferred to reporting zero and following the equilibrium strategy (obviously, the two reports are identical when realized sales is zero), it follows that a firm never finds it optimal to submit a report and then not make the corresponding payment.[32]

In sum, if (1) holds then, by choosing δ sufficiently close to one, the lysine strategy profile is a semi-public perfect equilibrium. Finally, for any $\varepsilon > 0$, when δ is sufficiently close to one,

$$\max\{\phi(m) : \underline{m} \leq m \leq \overline{m}\} = \beta(\overline{m} - \underline{m})(1-\delta) < \varepsilon,$$

so the probability of a price war in any given period can be made arbitrarily small.

9.2 Proof of Theorem 3

The way in which we will proceed is to consider a less constrained problem with a strict subset of the ICC and feasibility constraints. Once the mechanism is characterized, we'll show that the remaining ICC and feasibility constraints are satisfied. Specifically, we seek to maximize

$$\begin{aligned} &\max_{|t(r_1,r_2)|\leq x} \rho_0 t(0,0) + (\rho_1/2)[t(1,0)+t(0,1)] \\ &+\left(\frac{\rho_2}{4}\right)[t(2,0)+2t(1,1)+t(0,2)] \\ &+\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right)+\left(\frac{\rho_1}{2}\right)[t(0,1)-t(1,0)] \\ &+\left(\frac{\rho_2}{2}\right)[t(0,2)-t(2,0)] \end{aligned} \tag{44}$$

subject to these constraints:

$$t(2,0) \geq t(1,0) \tag{45}$$

$$t(2,0) \geq t(0,0) \tag{46}$$

$$\eta t(1,1) + (1-\eta)t(1,0) \geq \eta t(0,1) + (1-\eta)t(0,0) \tag{47}$$

$$0 \geq t(0,2) + t(2,0) \tag{48}$$

$$0 \geq t(1,1) \tag{49}$$

[32] It can also be shown that, if firms are sufficiently patient and reports are bounded, it is always optimal to pay zr after reporting r. Since monitoring is perfect, the usual argument works.

$$0 \geq t(0,1) + t(1,0) \tag{50}$$

$$0 \geq t(0,0) \tag{51}$$

(45)-(47) are the ICCs ensuring that a firm does not want to under-report its sales. $\eta \equiv \eta_{1,1}(p,p)$ so that (47) is (17) when evaluated at equilibrium prices. (48)-(51) are the feasibility constraints for when aggregate sales reports do not exceed 2.

The problem is then to choose $t(0,0), t(1,0), t(0,1), t(2,0), t(1,1),$ and $t(0,2)$ to maximize (44) subject to (45)-(51). Note that (44) is increasing in $t(1,1)$ and that $t(1,1)$ enters only (47) and (49). A higher value increases the maximand and loosens (47). Hence, (49) must be binding. Optimality then requires:

$$t(1,1) = 0. \tag{52}$$

Next note that (44) is increasing in $t(0,2)$ and that $t(0,2)$ enters only (48). If $x > t(0,2)$ then optimality requires (48) to bind:

$$t(2,0) + t(0,2) = 0. \tag{53}$$

If $x = t(0,2)$ then, by (48), it follows that $t(2,0) = -x$. Again, $t(2,0) + t(0,2) = 0$. Optimality then requires (53).

Using (52)-(53), defining $s = t(0,2) = -t(2,0)$, and simplifying, we can re-state (44) as choosing $t(0,0), t(1,0), t(0,1)$, and s (all in $[-x, x]$) to maximize:

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s \tag{54}$$

subject to

$$-s \geq t(1,0) \tag{55}$$

$$-s \geq t(0,0) \tag{56}$$

$$(1-\eta)\, t(1,0) \geq \eta t(0,1) + (1-\eta)\, t(0,0) \tag{57}$$

$$0 \geq t(0,1) + t(1,0) \tag{58}$$

$$0 \geq t(0,0) \tag{59}$$

Suppose (57) was not binding:

$$(1-\eta)\, t(1,0) > \eta t(0,1) + (1-\eta)\, t(0,0).$$

Even if (58) is binding, we can raise $t(0,1)$ and lower $t(1,0)$ (note that (55) will still be satisfied) so as to satisfy (58) and, because (54) is increasing in $t(0,1)$, the payoff is higher. The only caveat to the preceding argument is if $t(0,1) = x$, in which case $t(0,1)$ cannot be increased. But then, by (58), it follows that $t(1,0) = -x$. In that case, (57) takes the form:

$$\begin{aligned} (1-\eta)\, t(1,0) &\geq \eta t(0,1) + (1-\eta)\, t(0,0) \Leftrightarrow \\ -(1-\eta)\, x &\geq \eta x + (1-\eta)\, t(0,0) \Leftrightarrow \\ -\frac{x}{1-\eta} &\geq t(0,0) \end{aligned}$$

which is a contradiction since $t(0,0) \geq -x$. Hence, (57) must be binding:

$$(1-\eta)\, t(1,0) = \eta t(0,1) + (1-\eta)\, t(0,0) \Leftrightarrow$$

$$t(1,0) = t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1), \tag{60}$$

where, using Bayes Rule,

$$\eta = \frac{\rho_2 \psi(1;2)}{\rho_2 \psi(1;2) + \rho_1 \psi(1;1)} = \frac{\rho_2 (1/2)}{\rho_2 (1/2) + \rho_1 (1/2)} = \frac{\rho_2}{\rho_1 + \rho_2}.$$

Using (60) to substitute for $t(1,0)$ in (54), the problem is now to choose $t(0,0)$, $t(0,1)$, and s to maximize:

$$\left(\frac{2(1-\rho_0) - \rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s \tag{61}$$

subject to

$$-s \geq t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1) \tag{62}$$

$$-s \geq t(0,0) \tag{63}$$

$$0 \geq t(0,0) + \left(\frac{\rho_1 + \rho_2}{\rho_1}\right) t(0,1) \tag{64}$$

$$0 \geq t(0,0) \tag{65}$$

If an optimum has $s < 0$ then, since (61) is increasing in s, it must be the case that (62) and/or (63) are binding. By (65), if (63) binds then $s \geq 0$ which is a contradiction. Hence, if $s < 0$ is optimal then it implies (62) binds, which means that

$$t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1) > 0.$$

Since then $t(0,1) > 0$, it follows that

$$t(0,0) + \left(\frac{\rho_1 + \rho_2}{\rho_1}\right) t(0,1) > 0$$

which violates (64). Therefore, it cannot be the case that $s < 0$. We conclude that an optimum must have $s \geq 0$.

Suppose $0 > t(0,1)$. Since (61) is increasing in $t(0,1)$ then one of the constraints must bind. It follows from $0 > t(0,1)$ and (65) that (64) does not bind. When $0 > t(0,1)$, (63) binds before (62) which implies (62) does not bind. Thus, neither of the constraints involving $t(0,1)$ bind which means (61) can be increased by raising $t(0,1)$. We conclude that $t(0,1) \geq 0$ at an optimum.

To summarize the properties of an optimum derived thus far:

$$\begin{aligned} t(1,0) &= t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1) \\ t(1,1) &= 0 \\ 0 &\geq t(0,0) \\ t(0,1) &\geq 0 \\ t(0,2) &= -t(2,0) = s \geq 0. \end{aligned}$$

$t(0,1) \geq 0$ implies that if (64) holds then (65) holds which makes (65) redundant; and if (62) holds then (63) holds which makes (63) redundant. The problem is then: choose s, $t(0,0)$, and $t(0,1)$ to maximize

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s$$

subject to

$$t(0,1) - s \geq t(0,0) + \left(\frac{\rho_1+\rho_2}{\rho_1}\right) t(0,1) \tag{66}$$

$$0 \geq t(0,0) + \left(\frac{\rho_1+\rho_2}{\rho_1}\right) t(0,1) \tag{67}$$

where (62) has been rearranged. First note that it is not an optimum for $t(0,1) - s > 0$. In that case, (67) implies (66) is not binding. Since $t(0,1) > s$ implies $s < x$, s can be increased which raises the objective while continuing to satisfy the constraints. Therefore, $t(0,1) - s \leq 0$. Hence, if (66) holds then (67) holds, and, at an optimum, $s \geq t(0,1)$.

Thus, the problem is: choose s, $t(0,0)$, and $t(0,1)$ to maximize

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s$$

subject to

$$\begin{aligned} 0 &\geq s + t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1) \\ s &\geq t(0,1) \geq 0 \\ 0 &\geq t(0,0) \end{aligned}$$

By including the constraint $s \geq t(0,1)$, we ensure that satisfaction of (66) implies (67) holds. Suppose the first constraint does not bind at the optimum. As the objective is increasing in s, it must be the case that $s = x$. Hence, the constraint becomes:

$$0 > x + t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1),$$

but this cannot hold since $t(0,0) \geq -x$ and $t(0,1) \geq 0$. We conclude that the constraint binds:

$$0 = s + t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1).$$

Therefore, the problem is: choose s, $t(0,0)$, and $t(0,1)$ to maximize

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s \tag{68}$$

subject to

$$0 = s + t(0,0) + \left(\frac{\rho_2}{\rho_1}\right) t(0,1) \tag{69}$$

$$s \geq t(0,1) \geq 0 \tag{70}$$

$$0 \geq t(0,0) \tag{71}$$

- Assume $\rho_2 > \rho_0, \rho_1$.

Suppose $t(0,0) > -x$. Since we've shown that, at an optimum, $t(0,1) \geq 0$ then $x > s$ by (69). But the objective can be increased by raising s by $\varepsilon > 0$ (which is possible since $s < x$) and lowering $t(0,0)$ by ε. The objective goes up by $(\rho_2 - \rho_0)\varepsilon > 0$ and, in addition, (69) still holds. Therefore, $t(0,0) = -x$.

We now have that, at an optimum, $t(0,0) = -x$ and we previously showed $s, t(0,1) \geq 0$. (69) is now

$$0 = s - x + \left(\frac{\rho_2}{\rho_1}\right) t(0,1).$$

Use this condition to substitute for s in (68):

$$\begin{aligned} & \left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) - \rho_0 x + \rho_1 t(0,1) + \rho_2 \left[x - \left(\frac{\rho_2}{\rho_1}\right) t(0,1)\right] \\ = & \left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + (\rho_2 - \rho_0) x - \left(\frac{\rho_2^2 - \rho_1^2}{\rho_1}\right) t(0,1). \end{aligned}$$

Substituting for s in (70), we get

$$s \geq t(0,1) \Leftrightarrow x - \left(\frac{\rho_2}{\rho_1}\right) t(0,1) \geq t(0,1) \Leftrightarrow \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) x \geq t(0,1).$$

The problem is then: choose $t(0,1)$ to maximize

$$\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right) + (\rho_2 - \rho_0) x - \left(\frac{\rho_2^2 - \rho_1^2}{\rho_1}\right) t(0,1) \tag{72}$$

subject to

$$\left(\frac{\rho_1}{\rho_1 + \rho_2}\right) x \geq t(0,1). \tag{73}$$

Since $\rho_2 > \rho_1$ then (72) is decreasing in $t(0,1)$. By the derived condition that $t(0,1) \geq 0$, an optimum has $t(0,1) = 0$. (Also note that since $t(0,0) = -x$ and $s \leq x$, (69) would be violated if $t(0,1) < 0$.) From $t(0,0) = -x$ and $t(0,1) = 0$, it follows from (69) that $s = x$.

If $\rho_2 > \rho_0, \rho_1$ then the solution is

$$\begin{aligned} t(0,0) &= -x \\ t(0,1) &= 0, t(1,0) = -x \\ t(0,2) &= x, t(2,0) = -x \\ t(1,1) &= 0 \end{aligned}$$

and the objective takes the value:

$$\begin{aligned} & \left(\frac{2(1-\rho_0) - \rho_1}{4\xi'(0)}\right) + \rho_0 t(0,0) + \rho_1 t(0,1) + \rho_2 s \\ = & \left(\frac{2(1-\rho_0) - \rho_1}{4\xi'(0)}\right) + (\rho_2 - \rho_0) x \end{aligned}$$

To complete the analysis, we need to ensure that the remaining ICC and feasibility constraints are satisfied. For that purpose, we extend the transfer function to encompass sales reports that sum to more than two.

$$\begin{aligned} t(0,0) &= -x \\ t(0,1) &= 0, t(1,0) = -x, \\ t(0,2) &= x, t(2,0) = -x \\ t(1,1) &= 0 \\ t(r_1, r_2) &= -x \text{ if } r_1 + r_2 > 2 \end{aligned} \tag{74}$$

Notice that all feasibility constraints are satisfied.

Referring back to the complete set of ICCs, the ones that we still need to verify are satisfied are, for all p_1,[33]

$$\begin{aligned} & \eta_{1,1}(p_1, \widehat{p}) t(1,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(1,0) \\ \geq \; & \eta_{1,1}(p_1, \widehat{p}) t(2,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(2,0) \end{aligned} \tag{75}$$

$$\begin{aligned} & \eta_{1,1}(p_1, \widehat{p}) t(1,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(1,0) \\ \geq \; & \eta_{1,1}(p_1, \widehat{p}) t(0,1) + \left(1 - \eta_{1,1}(p_1, \widehat{p})\right) t(0,0) \end{aligned} \tag{76}$$

$$\begin{aligned} & \eta_{0,2}(p_1, \widehat{p}) t(0,2) + \eta_{0,1}(p_1, \widehat{p}) t(0,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(0,0) \\ \geq \; & \eta_{0,2}(p_1, \widehat{p}) t(1,2) + \eta_{0,1}(p_1, \widehat{p}) t(1,1) \\ & + \left(1 - \eta_{0,2}(p_1, \widehat{p}) - \eta_{0,1}(p_1, \widehat{p})\right) t(1,0) \end{aligned} \tag{77}$$

[33] Actually, we have already verified that (76) holds for $p_1 = \widehat{p}$.

$$
\begin{aligned}
& \eta_{0,2}(p_1,\widehat{p})\,t(0,2)+\eta_{0,1}(p_1,\widehat{p})\,t(0,1) \\
& +\left(1-\eta_{0,2}(p_1,\widehat{p})-\eta_{0,1}(p_1,\widehat{p})\right)t(0,0) \\
\geq\; & \eta_{0,2}(p_1,\widehat{p})\,t(2,2)+\eta_{0,1}(p_1,\widehat{p})\,t(2,1) \\
& +\left(1-\eta_{0,2}(p_1,\widehat{p})-\eta_{0,1}(p_1,\widehat{p})\right)t(2,0)
\end{aligned} \tag{78}
$$

Substituting (74) into (75),

$$
-\left(1-\eta_{1,1}(p_1,\widehat{p})\right)x \geq -\eta_{1,1}(p_1,\widehat{p})\,x-\left(1-\eta_{1,1}(p_1,\widehat{p})\right)x \Leftrightarrow \eta_{1,1}(p_1,\widehat{p}) \geq 0,
$$

which holds. Next consider (76):

$$
-\left(1-\eta_{1,1}(p_1,\widehat{p})\right)x \geq -\left(1-\eta_{1,1}(p_1,\widehat{p})\right)x.
$$

Next consider (77):

$$
\eta_{0,2}(p_1,\widehat{p})\,x-\left(1-\eta_{0,2}(p_1,\widehat{p})-\eta_{0,1}(p_1,\widehat{p})\right)x \geq -\eta_{0,2}(p_1,\widehat{p})\,x-\left(1-\eta_{0,2}(p_1,\widehat{p})-\eta_{0,1}(p_1,\widehat{p})\right)x \Leftrightarrow
$$

$$
\eta_{0,2}(p_1,\widehat{p}) \geq -\eta_{0,2}(p_1,\widehat{p}).
$$

Finally, consider (78):

$$
\eta_{0,2}(p_1,\widehat{p})\,x-\left(1-\eta_{0,2}(p_1,\widehat{p})-\eta_{0,1}(p_1,\widehat{p})\right)x \geq -x \Leftrightarrow 2\eta_{0,2}(p_1,\widehat{p})+\eta_{0,1}(p_1,\widehat{p}) \geq 0.
$$

We conclude that if $\rho_2 > \rho_0, \rho_1$ then (74) is an optimal mechanism.

- Assume $\rho_0 > \rho_1, \rho_2$.

Return to (68) with constraints (69)-(71). Suppose $t(0,0)=0$. Since we've already shown that, at an optimum, $s, t(0,1) \geq 0$, then (69) implies $s=0=t(0,1)$. Let us see if there is a better solution. Thus, suppose $t(0,0)<0$. $t(0,0)<0$ and (69) imply $t(0,1)>0$ and/or $s>0$. If $t(0,1)>0$ then (70) implies $s>0$. Hence, at an optimum, if $t(0,0)<0$ then $s>0$. If (70) is not binding - specifically, if $s>t(0,1)$ - then (68) can be increased by reducing s by ε and raising $t(0,0)$ by ε; the objective goes up by $(\rho_0-\rho_2)\,\varepsilon>0$ and (69) still holds. Given then that $s=t(0,1)$, the problem is to choose $t(0,0)$ and s to maximize

$$
\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right)+\rho_0 t(0,0)+(\rho_1+\rho_2)\,s
$$

subject to

$$
0=t(0,0)+\left(\frac{\rho_1+\rho_2}{\rho_1}\right)s.
$$

Substituting this constraint into the objective, the problem is to choose $t(0,0)$ and s to maximize

$$
\left(\frac{2(1-\rho_0)-\rho_1}{4\xi'(0)}\right)+(\rho_1+\rho_2)\left(\frac{\rho_1-\rho_0}{\rho_1}\right)s \tag{79}
$$

subject to

$$
0=t(0,0)+\left(\frac{\rho_1+\rho_2}{\rho_1}\right)s. \tag{80}
$$

Since $\rho_1 - \rho_0 < 0$ then (79) is decreasing in s. Given (80), $t(0,0)$ should be set as high as possible, which implies $t(0,0) = 0$ and, therefore, $s = 0$. The best solution is then:

$$t(0,0) = 0, t(0,1) = 0, t(1,0) = 0, t(0,2) = 0, t(2,0) = 0, t(1,1) = 0.$$

Hence, if $\rho_0 > \rho_1, \rho_2$ then no collusion can be sustained.

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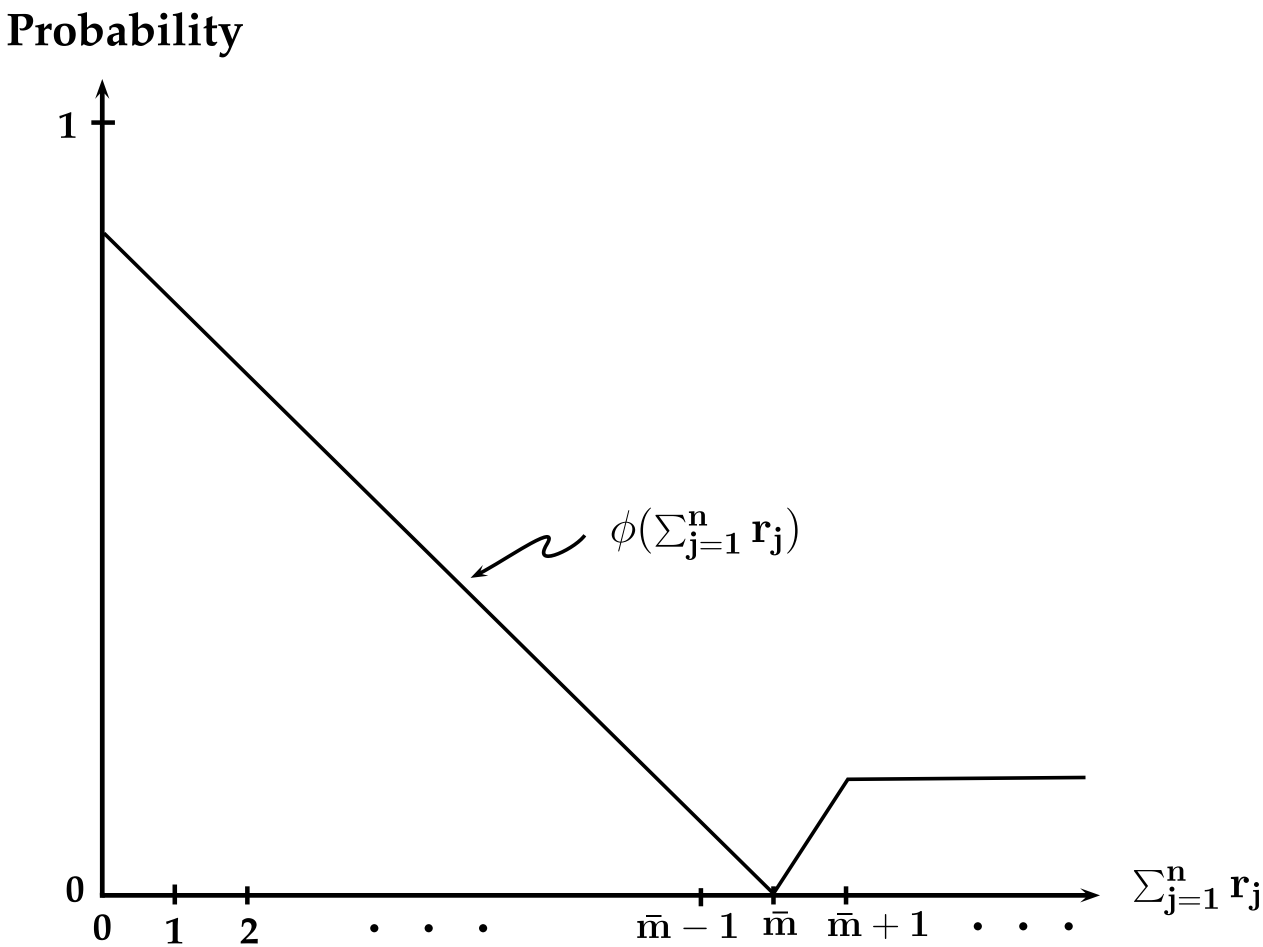
Probability
1
0
$\phi(\sum_{j=1}^{n} r_j)$
$\sum_{j=1}^{n} r_j$
0
1
2
. . .
$\bar{m}-1$
$\bar{m}$
$\bar{m}+1$
. . .