

Subjective Probabilities on a State Space

Edi Karni*

Johns Hopkins University

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Abstract

This paper extends the work of Karni (2009) in two distinct directions. First, it generalizes the model allowing for action-bet interaction and, consequently, the possibility that the decision-maker's risk attitudes may be affected by his choice of action. Second, it extends the analytical framework to include a state space and advances a choice-based definition of subjective probabilities that represent the beliefs of Bayesian decision makers regarding the likelihoods of events, thus resolving a fundamental difficulty with the definition of subjective probabilities.

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1 Introduction

Beginning with Ramsey (1927) and de Finetti (1937) and culminating with Savage (1954), the concept of choice-based subjective probabilities has been a subject of much interest and some controversy. With rare exceptions, subjective probabilities were treated as an aspect of the representation of a decision maker's preference relation, which *defines* his beliefs regarding the likelihood of events.¹

A radically different approach considers the decision maker's beliefs to be a cognitive phenomenon that feeds into the decision-making process. According to this approach the subjective probabilities *measure*, rather than define, the decision maker's beliefs.

A fundamental difficulty with the former approach is that the uniqueness of the subjective probabilities requires the use of some normalization, which is not implied by the structure of the preference relations and, consequently, is devoid of choice-theoretic meaning. For example, the definitions of subjective probabilities in the works of Savage (1954), Anscombe and Aumann (1963), Machina and Schmeidler (1992, 1995), and Gilboa and Schmeidler (1989), are based on the convention that constant acts (that is, functions that assign the same consequence to every state) yield the same utility in every state. This convention is not testable

¹This is the approach taken by Savage (1954) and all subsequent contributions invoking Savage's analytical framework. These include the subjective expected utility model of Anscombe and Aumann (1963), the probabilistic sophisticated choice theory of Machina and Schmeidler (1992, 1995), and the multi-prior model of Gilboa and Schmeidler (1989).

within the analytical framework of these models and must be taken on faith.² If the consequences correspond to state-contingent levels of wealth, then the subjective probabilities are defined by the marginal rates of substitution between state-contingent payoffs at certainty.³

A “measurement” approach to the definition of subjective probabilities was recently explored in Karni (2009). He dispenses with Savage’s notion of a state space, proposing instead a new analytical framework that consists of a set, Θ , of effects; a set, A , of actions (initiatives by which the decision maker believes he can affect the likelihoods of ensuing effects); a set, B , of bets on these effects; and a set of informative and not informative signals, \bar{X} , received before taking actions and choosing bets. The choice set, \mathcal{I} , consists of information-contingent plans (strategies) for choosing actions and bets (that is, a strategy is a function $I : \bar{X} \rightarrow A \times B$). In this framework, Karni (2009) develops a complete, choice-based, Bayesian decision theory in which decision makers’ preferences are represented by

$$I \mapsto \sum_{x \in \bar{X}} \left[\sum_{\theta \in \Theta} \pi(\theta, x | a_{I(x)}) [u(b_{I(x)}(\theta), \theta) + v(a_{I(x)})] \right] \quad (1)$$

where $a_{I(x)}$ and $b_{I(x)}$ are the action and bet assigned to the observation x by the strategy I ; $\{u(\cdot, \theta)\}_{\theta \in \Theta}$ are effect-dependent utility functions on the monetary payoffs of the bets; v is the (dis)utility of actions; and $\{\pi(\cdot, \cdot | a)\}_{a \in A}$ is a unique family of action-dependent, joint, subjective probabilities distributions on $\Theta \times \bar{X}$. The prior distributions $\{\pi(\cdot | o, a)\}_{a \in A}$ and the posterior distributions $\{\pi(\cdot | x, a)\}_{a \in A}$ on Θ are linked by Bayes rule and represent the

²In the same vein, Karni’s (1993) definition of subjective probability with state-dependent preferences uses the boundedness of the utility function to obtain the required normalization. Karni and Schmeidler (1993) use the marginal rates of substitution among payoffs in different states to normalize the utility functions.

³See Karni Schmeidler (1993) and Nau (1995) .

decision maker's prior and posterior beliefs.

This paper extends the work of Karni (2009) in two directions. First, it generalizes the model, allowing action-bet interaction and, consequently, the possibility that the decision maker's risk attitudes may be affected by his choice of action. In other words, the representation (1) is replaced by the more general form

$$I \mapsto \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta, x \mid a_{I(x)}) \tilde{u}(a_{I(x)}, b_{I(x)}(\theta), \theta), \quad (2)$$

where the utility functions $\{\tilde{u}(a, b(\theta), \theta)\}_{\theta \in \Theta}$ are not necessarily additive over actions and the bets. This generalization is of interest because it means that the model can be applied to the analysis of decision involving actions that have monetary dimensions (e.g., protecting a property against theft by installing an alarm system). It requires two main changes: the axiom of independent betting preferences of Karni (2009) is weakened to include action-dependent betting preferences and a new concept—strings of constant-utility bets—is introduced and incorporated into the analysis.

Second, this paper extends the analytical framework of Karni (2009) to include a state space and defines a unique, choice-based, subjective probability measure on the state space that generates the family of joint probability distributions $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ that figure in the representation (??). The uniqueness of the subjective probability measure on the state space resolves the fundamental difficulty with the definition of subjective probabilities mentioned above.

The paper consists of two main parts. Section 2 describes the analytical framework,

the preference structure, and the representation theorem. Section 3 defines the state space and characterizes the unique probability measure on this space that generates the family of action-dependent subjective probability distributions on the effects introduced in Section 2. Section 4 includes concluding remarks. The proofs are collected in the appendix.

2 The Model

2.1 The analytical framework

Following Karni (2009), let Θ be a finite set of *effects*; let A be a connected separable topological space, whose elements are referred to as *actions*; and let X a finite set of *observations* and denote by o the event that no observation materializes and define $\bar{X} = X \cup \{o\}$. A *bet* is a real-valued mapping on Θ , interpreted as monetary payoffs contingent on the realized effect. Let B denote the set of all bets and assume that it is endowed with the $\mathbb{R}^{|\Theta|}$ topology. Denote by $b_{-\theta}r$ the bet obtained from $b \in B$ by replacing the θ -coordinate of b , $b(\theta)$, with r .

Informative and noninformative signals in the form of observation may be received by the decision maker before he chooses a bet and an action, and they may affect his choice. The decision maker is supposed to formulate a strategy specifying the action-bet pairs to be implemented contingent on the observations. Formally, a *strategy* is a function $I : \bar{X} \rightarrow A \times B$ whose interpretation is a set of instructions specifying, for each informational event an action-bet pair, $I(x)$, to be implemented if the informational event x obtains. Let \mathcal{I} denote the

set of all strategies.

A decision maker is characterized by a preference relation \succsim on \mathcal{I} . The strict preference relation, \succ , and the indifference relation, \sim , are the asymmetric and symmetric parts of \succsim , respectively.

As usual, a consequence depicts those aspects of the decision problem that affect the decision maker's ex-post well-being. In this model, a *consequence* is a triplet (a, r, θ) representing, respectively, the action, the monetary payoff of the bets, and the effect. The set of all consequences is given by the Cartesian product $C = A \times \mathbb{R} \times \Theta$.

A *state of nature*, or a *state*, is a complete resolution of uncertainty, “a description of the world so complete that, if true and known, the consequences of every action would be known” (Arrow [1981], p. 45). Thus a state s is a function from \mathcal{I} to C . The set $S := \{s : \mathcal{I} \rightarrow C\}$ is the *state space*. Subsets of S are *events*. One of the elements of S is the *true state*. An event is said to *obtain* if the true state is an element of it.

Uncertainty in this model is resolved in two stages. In the interim stage, an observation, $x \in \bar{X}$, obtains and the action and bets prescribed by the strategies for that observation are implemented. In the second stage, the effect is realized and the payoffs of the bet are made. Let Ω be the set of all functions from the set of action to the set of effects (that is, $\Omega := \{\omega : A \rightarrow \Theta\}$). Elements of Ω depict the resolution of uncertainty surrounding the effects. Thus $S = \bar{X} \times \Omega$, and each state $s = (x, \omega)$ is an intersection of an *informational event* $\{x\} \times \Omega$ and a *material event* $\bar{X} \times \{\omega\}$. In other words, a state has two distinct dimensions

corresponding to the two stages of the resolution of uncertainty, the purely informational dimension, x , and the possibly substantive dimension, ω . The informational event does not affect the decision maker's well-being directly whereas the material event may.

In general, states are abstract representations of the resolution of uncertainty. In some situations, however, it is natural to attribute a concrete interpretation to the states. In the following examples, the informational events are weather forecasts, and the material events correspond to specific physical phenomena.

Example 1 *A passenger who wants to get from here to there must choose among driving, taking a bus, or flying (Luce and Krantz 1971). Whether and when he arrives at his destination depends on conditions beyond his control, such as the weather, the mechanical functioning of the alternative means of transportation, road congestion, and so forth. Conceivably, before choosing the means of transportation and placing a bet on the outcome (for example, by taking out insurance), the passenger may get some relevant information (for example, weather forecast) that may affect his decision. Uncertainty regarding the outcome of the trip is resolved once the weather forecast is obtained and the mechanical functioning of the alternative means of transportation, roads congestion, and so forth become known.*

Example 2 *Faced with the prospect of an approaching hurricane, a decision maker must make a plan that, contingent on the weather reports, may include boarding up his house, moving his family to a shelter, and betting on the damage (that is, taking out insurance). The uncertainty is resolved once the weather forecast is obtained, the plan is put into effect,*

the storm passes, and its path and force have been determined.

A main concern of this paper is to define a σ -algebra, \mathcal{E} , on S and a unique probability measure, P , on the measurable space (S, \mathcal{E}) such that (a) the conditioning of P on the noninformative signal o represents the decision maker's prior beliefs and (b) the conditioning of P on informative signals $x \in X$ represent the decision maker's posterior beliefs.

Denote by $I_{-x}(a, b) \in \mathcal{I}$ the strategy in which the x coordinate of $I, I(x)$, is replaced by (a, b) . The truncated strategy I_{-x} is referred to as a substrategy. For every given $x \in \bar{X}$, denote by \succsim^x the induced preference relation on $A \times B$ defined by $(a, b) \succsim^x (a', b')$ if and only if $I_{-x}(a, b) \succsim I_{-x}(a', b')$. The induced strict preference relation, denoted by \succ^x , and the induced indifference relation, denoted by \sim^x , are the asymmetric and symmetric parts of \succsim^x , respectively.⁴ The induced preference relation \succsim^o is referred to as the *prior* preference relation; the preference relations $\succsim^x, x \in X$, are the *posterior* preference relations. An observation, x , is *essential* if $(a, b) \succ^x (a', b')$ for some $(a, b), (a', b') \in A \times B$. I assume throughout that all elements of \bar{X} are essential.

For every $a \in A$ and $x \in \bar{X}$, define a binary relation \succsim_a^x on B by $b \succsim_a^x b'$ if and only if $(a, b) \succsim^x (a, b')$, for all $b, b' \in B$. The asymmetric and symmetric parts of \succsim_a^x are denoted by \succ_a^x and \sim_a^x , respectively.

An effect, θ , is said to be *nonnull given the observation-action pair* (x, a) if $(b_{-\theta}r) \succ_a^x (b_{-\theta}r')$, for some $b \in B$ and $r, r' \in \mathbb{R}$; it is *null given the observation-action pair* (x, a)

⁴For preference relations satisfying (A.1) - (A.3) below, these relations are well-defined.

otherwise. Given a preference relation, \succsim , denote by $\Theta(a, x)$ the subset of effects that are nonnull given the observation-action pair (x, a) . Assume that $\Theta(a, o) = \Theta$, for all $a \in A$. Note that if $\omega(a) \notin \Theta(a, x)$, then the state (x, ω) is null.

2.2 The preference structure

With slight variations in axioms (A.4), (A.6), and (A.7), all the axioms below were introduced, and their meaning discussed, in Karni (2009). I therefore refrain from further elaboration here.

Consider the following axioms depicting the structure of a preference relation \succsim on \mathcal{I} .

(A.1) (**Weak order**) \succsim is a complete and transitive binary relation.

A topology on \mathcal{I} is needed to define continuity of the preference relation \succsim . Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$, and let \mathcal{I} be endowed with the product topology.⁵

(A.2) (**Continuity**) For all $I \in \mathcal{I}$, the sets $\{I' \in \mathcal{I} \mid I' \succsim I\}$ and $\{I' \in \mathcal{I} \mid I \succsim I'\}$ are closed.

The next axiom, coordinate independence, is analogous to but weaker than Savage's (1954) sure-thing principle.⁶

⁵That is, the topology on \mathcal{I} is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

⁶See Wakker (1989) for details.

(A.3) (**Coordinate independence**) For all $x \in \bar{X}$, $I, I' \in \mathcal{I}$, and $(a, b), (a', b') \in A \times B$,
 $I_{-x}(a, b) \succ I'_{-x}(a, b)$ if and only if $I_{-x}(a', b') \succ I'_{-x}(a', b')$.

The next axiom requires that the “intensity of preferences” for monetary payoffs contingent on any given effect be independent of the observation. It is a weakening of axiom (A.4) in Karni (2009), which required, in addition, that the effect-contingent “intensity of preferences” for monetary payoffs be independent of the actions.

(A.4) (**Observation-independent action-betting preferences**) For all $x, x' \in \bar{X}$, $b, b', b'', b''' \in B$, $\theta \in \Theta(x) \cap \Theta(x')$, and $r, r', r'', r''' \in \mathbb{R}$, if $(a, b_{-\theta}r) \succ^x (a, b'_{-\theta}r')$, $(a, b'_{-\theta}r'') \succ^x (a, b_{-\theta}r''')$, and $(a, b''_{-\theta}r') \succ^{x'} (a, b'''_{-\theta}r)$ then $(a, b''_{-\theta}r'') \succ^{x'} (a, b'''_{-\theta}r''')$.

To link the decision maker’s prior and posterior probabilities, the next axiom asserts that in and of itself, information is worthless. That is, given an action, the preferences on bets conditional on there being no new information is the same as that when new information may not be used to select the bet. To state this axiom, I introduce the following additional notation: Let $I^{-o}(a, b)$ denote the strategy that assigns the action-bet pair (a, b) to every observation other than o (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$).

(A.5) (**Belief consistency**) For every $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, $I_{-o}(a, b) \sim I_{-o}(a, b')$ if and only if $I^{-o}(a, b) \sim I^{-o}(a, b')$.

2.3 Strings of constant-utility bets

Bets whose payoffs offset the direct impact of the effects are *constant-utility bets*. Because of the weakening of (A.4), unlike in Karni (2009), in this paper the constant utility bets are not independent of the actions. This requires a modification of the analysis and a new concept, dubbed *strings of constant utility bets*.

Definition 3 A mapping $\bar{b} : A \rightarrow B$ is a **string of constant-utility bets according to \succsim** if, for all $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$, $I_{-x}(a'', \bar{b}(a'')) \sim I'_{-x}(a''', \bar{b}(a'''))$ and $I''_{-x'}(a, \bar{b}(a)) \sim I'''_{-x'}(a', \bar{b}(a'))$ imply $I''_{-x'}(a'', \bar{b}(a'')) \sim I'''_{-x'}(a''', \bar{b}(a'''))$ and $\cap_{x \in X} \{b \in B \mid b \sim_a^x \bar{b}(a)\} = \{\bar{b}(a)\}$, for all $a \in A$.

To render the definition meaningful it is assumed that, given a string of constant-utility bets \bar{b} , for all $a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$ there are $I, I', I'', I''' \in \mathcal{I}$ such that the indifferences $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$, $I_{-x}(a'', \bar{b}(a'')) \sim I'_{-x}(a''', \bar{b}(a'''))$ and $I''_{-x'}(a, \bar{b}(a)) \sim I'''_{-x'}(a', \bar{b}(a'))$ hold.

Let $\mathcal{B}(\succsim)$ denote the set of all strings of constant-utility bets according to \succsim .

Definition 4 The set $\mathcal{B}(\succsim)$ is **inclusive** if there is $\bar{b} \in \mathcal{B}(\succsim)$ such that $(a, b) \sim^x (a, \bar{b}(a))$, for every $x \in X$ and $(a, b) \in A \times B$.

If, for some actions, there exists no monetary compensation for the impact of the effects (that is, the ranges of the utility of the monetary payoffs across effects do not overlap), then,

for that action, there is no constant utility bet and $\mathcal{B}(\succsim)$ is empty. Here I am concerned with the case in which $\mathcal{B}(\succsim)$ is inclusive, and thus nonempty.

In the special case $I = I'$ and $I'' = I'''$, definition 3 implies that $(a, \bar{b}(a)) \sim_{-x} (a', \bar{b}(a'))$ for all $x \in \bar{X}$. In other words, given a string of constant-utility bets, preferences among actions are observation independent. Anticipating the main result, this means that $(a, \bar{b}(a))$ and $(a', \bar{b}(a'))$ correspond to the same expected utility, regardless of the observation.⁷ This special case pertains, naturally, to actions identified with monetary expenses that are perfect substitutes for the payoffs of the bets. In general, however, it is possible that there are no feasible monetary compensation for the disutility associated with some actions. In such a case, the expected utilities associated with $(a, \bar{b}(a))$ and $(a', \bar{b}(a'))$ are distinct, and the difference between them is “measured” by the utility differences between the substrategies I_{-x} and I'_{-x} as well as those between I''_{-x} and I'''_{-x} .

The next axiom requires that the trade-offs between the actions and the substrategies that figure in definition 3 be independent of the constant-utility bets.

(A.6) (**Trade-off independence**) For all $I, I' \in \mathcal{I}$, $x \in \bar{X}$, $a, a' \in A$ and $\bar{b}, \bar{b}' \in \mathcal{B}(\succsim)$, $I_{-x}(a, \bar{b}(a)) \succsim I'_{-x}(a', \bar{b}(a'))$ if and only if $I_{-x}(a, \bar{b}'(a)) \succsim I'_{-x}(a', \bar{b}'(a'))$.

Finally, it is also required that the direct effect (that is, the cost) of actions, measured by the preferential difference between any two strings of constant-utility bets, $\bar{b}, \bar{b}' \in \mathcal{B}(\succsim)$, be independent of observation.

⁷I thank Jacques Drèze for calling my attention to this special case.

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b}' \in \mathcal{B}(\succsim)$, $x, x' \in \bar{X}$, and $a, a' \in A$, $(a, \bar{b}(a)) \succsim^x (a', \bar{b}'(a'))$ if and only if $(a, \bar{b}(a)) \succsim^{x'} (a', \bar{b}'(a'))$.

2.4 Representation

The next theorem generalizes Theorem 2 of Karni (2009) by permitting interaction between actions and bets. Consequently, the effect-dependent utility functions are not necessarily separately additive in actions and bets. To simplify the statement of the results that follow, I let $I(x) = (a_{I(x)}, b_{I(x)})$.

Theorem 5 *Let \succsim be a preference relation on \mathcal{I} and suppose that $\mathcal{B}(\succsim)$ is inclusive, then*

(a) *The following conditions are equivalent:*

(i) *\succsim satisfies (A.1)–(A.7).*

(ii) *there exist a continuous, real-valued function \tilde{u} on $A \times \mathbb{R} \times \Theta$, and a family of joint probability measures $\{\pi(\cdot, \cdot | a)\}_{a \in A}$ on $\bar{X} \times \Theta$ such that \succsim on I is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \sum_{\theta \in \Theta} \tilde{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) \pi(\theta | x, a_{I(x)}) \quad (3)$$

where $\mu(x) = \sum_{\theta \in \Theta} \pi(x, \theta | a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A$, $\pi(\theta | x, a) = \pi(x, \theta | a) / \mu(x)$ for all $x \in X$, and $\pi(\theta | o, a) = \frac{1}{1 - \mu(o)} \sum_{x \in X} \pi(x, \theta | a)$.

(b) *The function \tilde{u} is unique up to a positive affine transformation and, for each $a \in A$, $\pi(\cdot, \cdot | a)$ is unique.*

(c) For every $\bar{b} \in B(\succ)$ and $a \in A$, $\tilde{u}(a, \bar{b}(a)(\theta), \theta) = \tilde{u}(a, \bar{b}(a)(\theta'), \theta')$ for all $\theta, \theta' \in \Theta$.

Notice that, although the joint probability distributions $\pi(\cdot, \cdot | a)$, $a \in A$ depend on the actions, the distribution μ is independent of a . This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information arrives, it precedes the choice of action. Hence the dependence of the joint probability distributions $\pi(\cdot, \cdot | a)$ on a captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.⁸

A special case of Theorem 5 obtains when actions are monetary expenditures (that is, when $A = \mathbb{R}_-$). For instance, when considering installing sprinklers to reduce the loss in case of a fire, it is natural to assume that the utility impact of this action depends solely on the money spent. Hence $\tilde{u}(a, b(\theta), \theta) = \tilde{u}(a + b(\theta), \theta)$, $\theta \in \Theta$. In general, actions affect the preference directly, through their associated disutility and the possible associated “wealth effect” on the decision maker's attitudes toward the risk represented by the bets, and indirectly, through their impact on the probabilities of the effects. To isolated the “utility impact,” it is necessary to confine attention to strings of constant-utility bets. The idea

⁸If an action-bet pair are already “in effect” when new information arrives, they constitute a default course of action. In such instance, the interpretation of the decision at hand is possible choice of new action and bet. For example, a modification, upon learning the results of medical checkup, of a diet regiment coupled with a possible change of life insurance policy.

that, insofar as the utility is concerned, actions and bets are perfect substitutes is captured by the following axiom:

(A.8) **(Substitution)** For all $\bar{b} \in \mathcal{B}(\succsim)$, $I \in \mathcal{I}$, $x \in \bar{X}$ and $a \in \mathbb{R}_-$, $z \in \mathbb{R}$, $I_x(a, \bar{b}(a)) \sim I_x(a - z, \bar{b}(a) + z)$.

By Theorem 5 and axiom (A.8), for every $\bar{b} \in \mathcal{B}(\succsim)$ and $a \in A$, $\tilde{u}(a, \bar{b}(a)(\theta), \theta) = \tilde{u}(a - z, \bar{b}(a)(\theta) + z, \theta)$ for all $\theta \in \Theta$. Hence, with slight abuse of notations, $\tilde{u}(a, \bar{b}(a)(\theta), \theta) = \tilde{u}(a + \bar{b}(a)(\theta), \theta)$ for $\theta \in \Theta$. This implies the following:

Corollary 6 *Let $A = \mathbb{R}_-$ and \succsim be a preference relation on \mathcal{I} and suppose that $\mathcal{B}(\succsim)$ is inclusive. Then \succsim satisfies (A.1)–(A.8) if and only if there exist a continuous, real-valued function \tilde{u} on $A \times \mathbb{R} \times \Theta$, and a family of joint probability measures $\{\pi(\cdot, \cdot | a)\}_{a \in A}$ on $\bar{X} \times \Theta$ such that \succsim on \mathcal{I} is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \sum_{\theta \in \Theta} \tilde{u}(a_{I(x)} + b_{I(x)}(\theta), \theta) \pi(\theta | x, a_{I(x)}) \quad (4)$$

where $\mu(x) = \sum_{\theta \in \Theta} \pi(x, \theta | a)$ for all $x \in \bar{X}$ is independent of a and, for each $a \in A$, $\pi(\theta | x, a) = \pi(x, \theta | a) / \mu(x)$ for all $x \in X$, and $\pi(\theta | o, a) = \frac{1}{1 - \mu(o)} \sum_{x \in X} \pi(x, \theta | a)$. Moreover, \tilde{u} is unique up to positive affine transformation, for each $a \in A$ $\pi(\cdot, \cdot | a)$ is unique and, for every $\bar{b} \in \mathcal{B}(\succsim)$ and $a \in A$, $\tilde{u}(a + \bar{b}(a)(\theta), \theta) = \tilde{u}(a + \bar{b}(a)(\theta'), \theta')$ for all $\theta, \theta' \in \Theta$.

3 Subjective Probabilities on the State Space

3.1 Action-dependent subjective probabilities on S

Let \mathcal{Y} be the informational partition of S (that is, mutually disjoint informational events whose union is S). Formally, $\mathcal{Y} = \{\{x\} \times \Omega \mid x \in \bar{X}\}$.

Define a function $t : A \times \Omega \rightarrow \Theta$ by $t(a, \omega) = \omega(a)$. This function may be interpreted as a technology that determines the realized effect as a result of the actions (for example, time, effort, money spent) taken by the decision maker and the material aspects of the true state of nature. For every $a \in A$ and $\theta \in \Theta$, let $\mathcal{T}_a(\theta) := t^{-1}(\theta, a) \subseteq \Omega$. Then $\mathcal{T}_a = \{\mathcal{T}_a(\theta) \mid \theta \in \Theta\}$ is a (finite) partition of Ω . Denote by \mathcal{E}_a the σ -algebra on S generated by $\mathcal{Y} \wedge \mathcal{T}_a$, the join of \mathcal{Y} and \mathcal{T}_a .⁹ Elements of \mathcal{E}_a are *events*. Hence events are unions of elements of $\mathcal{Y} \wedge \mathcal{T}_a$. Consider the measurable spaces (S, \mathcal{E}_a) , $a \in A$. By Theorem 5, a preference relation \succsim satisfying (A.1)–(A.7) implies the existence of a unique joint probability measure $\pi(\cdot, \cdot \mid a)$ on $\bar{X} \times \Theta$. For each $x \in X$ and $a \in A$, let η_a be a probability measure on \mathcal{E}_a defined by $\eta_a(E) = \sum_{x \in Z} \sum_{\theta \in \Upsilon(x)} \pi(x, \theta \mid a)$ for every $E = \cup_{x \in Z} \cup_{\theta \in \Upsilon(x)} t^{-1}(\theta, a)$, $Z \subseteq \bar{X}$ and $\Upsilon(x) \subseteq \Theta$. The uniqueness of the conditional probability measures follows from Theorem 5. By definition, the prior probability measure on Ω is given by $\eta_a(\{o\} \times t^{-1}(\Upsilon, a)) = \sum_{\theta \in \Upsilon} \pi(o, \theta \mid a)$, for all $\Upsilon \subseteq \Theta$. The posterior probability measures on Ω are given by $\eta_a(\{x\} \times t^{-1}(\Upsilon, a)) = \sum_{\theta \in \Upsilon} \pi(x, \theta \mid a)$, for all $\Upsilon \subseteq \Theta$ and $x \in X$. The uniqueness of the prior and posteriors

⁹The join of two partitions is the coarsest common refinement of these partitions.

follow from the uniqueness of π . Moreover, by Theorem 5, $\eta(\{x\} \times \Omega) = \sum_{\theta \in \Theta} \pi(x, \theta | a)$ is independent of a .

Theorem 5 may be restated in terms of these probability measures as follows:

Corollary 7 *Let \succsim be a preference relation on \mathcal{I} and suppose that $\mathcal{B}(\succsim)$ is inclusive. Then \succsim satisfies (A.1)-(A.7) if and only if there exist an array of continuous, real-valued function \tilde{u} on $A \times \mathbb{R} \times \Theta$ and, for each $a \in A$, a unique probability measures η_a on the measurable space (S, \mathcal{E}_a) such that \succsim on \mathcal{I} is represented by*

$$I \mapsto \sum_{x \in \bar{X}} \eta(\{x\} \times \Omega) \sum_{\theta \in \Theta} \tilde{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) \eta_{a_{I(x)}}(\{x\} \times t^{-1}(\theta, a_{I(x)})). \quad (5)$$

Moreover, \tilde{u} is unique up to positive affine transformation.

Corollary 7 asserts the existence of a unique family of action-dependent prior and posterior probability measures on the measurable spaces $\{(S, \mathcal{E}_a)\}_{a \in A}$. This collection of measure spaces is sufficiently rich to allow an action-dependent probability to be defined for every event that matters to the decision maker, given all the conceivable choices among strategies that he might be called upon to make. Hence from the point of view of Bayesian decision theory, the family of action-dependent subjective probability measures defined here is complete, in the sense of being well defined for every conceivable decision problem that can be formulated in this framework. However, there is no guarantee that these subjective probability measures are mutually consistent. Hence it interesting to inquire about the necessary and sufficient conditions for the existence of a unique probability space that supports all

these action-dependent measures in the sense that $\eta_a(E)$ coincides with this measure on the events $E \in \mathcal{E}_a$. This issue is taken up next.

3.2 Subjective probabilities on S

To begin with, it is necessary to define a σ -algebra that includes the class of events $\cup_{a \in A} \mathcal{E}_a$.¹⁰ Let $\bigwedge_{i=1}^k \mathcal{T}_{a_i}$ be the join of $\mathcal{T}_{a_1}, \dots, \mathcal{T}_{a_k}$, and denote by \mathcal{E} the σ -algebra generated by $\{\bigwedge_{i=1}^k \mathcal{T}_{a_i} \mid k < \infty\} \wedge \mathcal{Y}$. The issue posed at the end of the last subsection may be restated formally as follows: What are the necessary and sufficient conditions for the existence of a unique probability measure P on the measurable space (S, \mathcal{E}) , such that $P(E) = \eta_a(E)$ for all $E \in \cup_{a \in A} \mathcal{E}_a$, and $P(\{x\} \times \Omega) = \eta(\{x\} \times \Omega)$?

This question raises two issues. The first, mentioned above, concerns the consistency of beliefs across actions. To illustrate this issue and motivate the ensuing inquiry, I consider Example 2 of the preceding section. In this example, the substantive aspect of the state is concrete, namely, the force and path of the approaching hurricane. The effects of the hurricane are determined by the action taken, a , and the force and path of the hurricane, ω , according to the technology $t(a, \omega)$. In this case, the conditional probability of the effects conditional on the action, and the observation, $\pi(\theta, x \mid a)$ corresponds to probability $\eta_a(E)$, where $E = \{x\} \times t^{-1}(\theta, a)$. It is also intuitively clear that, if $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ then the probability that E obtains, (that is, that the true state consists of the forecast x and that the force and

¹⁰Note that $\cup_{a \in A} \mathcal{E}_a$ itself is not a σ -algebra.

trajectory of the hurricane is a point in the set $t^{-1}(\theta, a)$ are independent of the action. Formally, if $\{x\} \times t^{-1}(\theta', a') = \{x\} \times t^{-1}(\theta, a) = E$, then, because the probabilities of the effects are induced by the probabilities of the underlying state, consistency requires that $\pi(\theta, x | a) = \eta_a(E) = \eta_{a'}(E) = \pi(\theta', x | a')$.

The second issue arises because a probability measure on the measurable space (S, \mathcal{E}) assigns probabilities to nonempty intersections of the preimages, under t , of effects, for distinct actions. In symbols, it invokes the probabilities of events such as $t^{-1}(\theta, a) \cap t^{-1}(\theta', a') \neq \emptyset$ for some $\theta, \theta' \in \Theta$ and distinct actions $a, a' \in A$. In general, the probabilities of such events may not be determined from the probabilities of the events in the collection $\cup_{a \in A} \mathcal{E}_a$.

These issues are addressed in next two subsections.

3.2.1 Belief consistency

For each $a \in A$, $\bar{b}(a), \bar{b}'(a) \in \mathcal{B}$, and $E \in \mathcal{E}_a$, let $\bar{b}(a)_E \bar{b}'(a) \in B$ be defined by

$$\bar{b}(a)_E \bar{b}'(a)(t(a, \omega)) = \left\{ \begin{array}{ll} \bar{b}(a)(t(a, \omega)) & \text{if } \omega \in E \\ \bar{b}'(a)(t(a, \omega)) & \text{if } \omega \in \Omega - E \end{array} \right\}.$$

Thus $\bar{b}(a)_E \bar{b}'(a)$ is the bet whose utility payoffs is a doubleton, paying $\tilde{u}(a, \bar{b}(a))$ for all θ such that $t^{-1}(\theta, a) \in E$ and $\tilde{u}(a, \bar{b}'(a))$ for all θ such that $t^{-1}(\theta, a) \in \Omega - E$.

Definition 8 : For all $E \in \cup_{a \in A} \mathcal{E}_a$ and $a, a' \in A$, if $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ then E is **believed equally likely to obtain under a and under a'** given $x \in \bar{X}$, if $I_{-x}(a, \bar{b}(a)_E \bar{b}'(a)) \sim$

$I'_{-x}(a', \bar{b}(a')_E \bar{b}'(a'))$, where $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$, $I_{-x}(a, \bar{b}'(a)) \sim I'_{-x}(a', \bar{b}'(a'))$ and $I_{-x}(a, \bar{b}(a)) \succ I'_{-x}(a, \bar{b}'(a))$.

Let this (partial) binary relation on $\cup_{a \in A} \mathcal{E}_a$ be denoted by $(a, E) \sim_L^x (a', E)$, then \sim_L^x is an equivalence. The axiom requires that if the same event may obtain under two distinct actions, then the beliefs about its likelihood be action independent. Formally,

(A.9) (**Belief consistency**) For all $a, a' \in A$ and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$, $(a, E) \sim_L^x (a', E)$.

The following theorem asserts that, under belief consistency, the action-dependent subjective probability measures, $\{\eta_a\}_{a \in A}$, agree on events in the intersections of the σ -algebras $\{\mathcal{E}_a\}_{a \in A}$.

Theorem 9 *Let \succ be a preference relation on \mathcal{I} satisfying (A.1)–(A.7), and suppose that $\mathcal{B}(\succ)$ is inclusive. Then \succ satisfies belief consistency, (A.9), if and only if $\eta_a(E) = \eta_{a'}(E)$, for all $a, a' \in A$ and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$.*

3.2.2 Richness

Consider next the problem posed by events in \mathcal{E} that are in the intersection of elements of the partition of the state space under distinct actions. In general, the probabilities of such events cannot be inferred from the action-dependent joint probabilities of observations and effects that figure in Theorem ???. This problem can be surmounted if the model is rich in

the sense that, for every finite collection of actions and corresponding subsets of effects, the event that yields these effects under these actions coincides with an event in \mathcal{E}_a for some $a \in A$. Formally,

Definition 10 *The model $(S, A, \Theta, \bar{X}, t)$ is **rich** if, for all $(a_i, \Upsilon_i) \in A \times 2^\Theta$, $i = 1, \dots, k$, and $x \in \bar{X}$, $\{x\} \times \cap_{i=1}^k t^{-1}(\Upsilon_i, a_i) = E$ for some $E \in \cup_{a \in A} \mathcal{E}_a$.*

The next theorem establishes the existence and uniqueness of a probability space supporting the action-dependent distributions.

Theorem 11 *Let \succsim be a preference relation on \mathcal{I} . Suppose that $\mathcal{B}(\succsim)$ is inclusive and that the model $(S, A, \Theta, \bar{X}, t)$ is rich. Then \succsim satisfies (A.1)–(A.7) and (A.9) if and only if there exist a continuous, real-valued function \tilde{u} on $A \times \mathbb{R} \times \Theta$ and a unique probability measure P on (S, \mathcal{E}) such that \succsim on \mathcal{I} is represented by*

$$I \mapsto \sum_{x \in \bar{X}} P(\{x\} \times \Omega) \sum_{\theta \in \Theta} \tilde{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) P(\{x\} \times t^{-1}(\theta, a_{I(x)})). \quad (6)$$

Moreover, \tilde{u} is unique up to positive affine transformation.

The proof invokes the probability distributions of Theorem 5 and the richness of the model to define a probability measure on the algebra generated by $\{\wedge_{i=1}^k \mathcal{T}_{a_i} \mid k < \infty\} \wedge \mathcal{Y}$. This probability measure has a unique extension to (S, \mathcal{E}) .

4 Concluding Remarks

Special instances of the theory presented here, including representations of preference relations displaying effect-independence preferences, may be obtained following Karni (2009).

In addition to providing choice-based foundations of the subjective probabilities, this paper furnishes an axiomatic foundation of the agent's behavior in the state-space formulation of the principal-agent problem.¹¹

¹¹See Hart and Holmstrom (1987)

APPENDIX

For expository convenience, I write \mathcal{B} instead of $\mathcal{B}(\succsim)$.

4.1 Proof of Theorem 5

(a) (Sufficiency) Assume that \succsim on \mathcal{I} satisfies (A.1)–(A.7) and \mathcal{B} is inclusive. Let \mathcal{I} be endowed with the product topology and suppose that $|\bar{X}| \geq 3$.¹²

By Wakker (1989) Theorem III.4.1, \succsim satisfies (A.1)–(A.3) if and only if there exist an array of real-valued functions $\{w(\cdot, \cdot, x) \mid x \in \bar{X}\}$ on $A \times B$ such that \succsim is represented by

$$I \mapsto \sum_{x \in \bar{X}} w(a_{I(x)}, b_{I(x)}, x), \quad (7)$$

where $w(\cdot, \cdot, x), x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.¹³

Since \succsim satisfies (A.4), Lemmas 5 and 6 in Karni (2006) and Theorem III.4.1 in Wakker (1989) imply that, for every $x \in \bar{X}$ and $a \in A$ such that $\Theta(a, x)$ contains at least two effects,

¹²To simplify the exposition I state the theorem for the case in which \bar{X} contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker [1989] theorem III.4.1).

¹³An array of real-valued functions $(v_s)_{s \in S}$ is said to be a *jointly cardinal additive representation* of a binary relation \succeq on a product set $D = \prod_{s \in S} D_s$ if, for all $d, d' \in D$, $d \succeq d'$ if and only if $\sum_{s \in S} v_s(d_s) \geq \sum_{s \in S} v_s(d'_s)$, and the class of all functions that constitute an additive representation of \succeq consists of those arrays of functions, $(\hat{v}_s)_{s \in S}$, for which $\hat{v}_s = \lambda v_s + \zeta_s$, $\lambda > 0$ for all $s \in S$. The representation is continuous if the functions $v_s, s \in S$ are continuous.

there exist array of functions $\{v_x(a, \cdot; \theta) : \mathbb{R} \rightarrow \mathbb{R} \mid \theta \in \Theta\}$ constituting a jointly cardinal, continuous, additive, representation of \succsim_a^x on B . Moreover, by the proof of Lemma 6 in Karni (2006), \succsim satisfies (A.1)–(A.4) if and only if, for every $x, x' \in \bar{X}$ and $a \in A$ satisfying $\Theta(a, x) \cap \Theta(a, x') \neq \emptyset$ and $\theta \in \Theta(a, x) \cap \Theta(a, x')$, there exist $\beta_{(x', x, a, \theta)} > 0$ and $\alpha_{(x', x, a, \theta)}$ such that $v_{x'}(a, \cdot, \theta) = \beta_{(x', x, a, \theta)} v_x(a, \cdot, \theta) + \alpha_{(x', x, a, \theta)}$.¹⁴

Define $u(a, \cdot, \theta) = v_o(a, \cdot, \theta)$, $\lambda(a, x; \theta) = \beta_{(x, o, a, \theta)}$ and $\alpha(a, x, \theta) = \alpha_{(x, o, a, \theta)}$ for all $a \in A$, $x \in \bar{X}$, and $\theta \in \Theta$. For every given $x \in \bar{X}$ and $a \in A$, $w(a, b, x)$ represents \succsim_a^x on B . Hence

$$w(a, b, x) = H \left(\sum_{\theta \in \Theta} (\lambda(a, x, \theta) u(a, b(\theta); \theta) + \alpha(a, x, \theta)), a, x \right), \quad (8)$$

where H is a continuous, increasing function.

Consider next the restriction of \succsim to $L := \{(a, \bar{b}(a)) \mid a \in A, \bar{b} \in \mathcal{B}\}^{\bar{X}}$.

Lemma 12 *There exist functions $U : L \rightarrow \mathbb{R}$, $\xi : \bar{X} \rightarrow \mathbb{R}_{++}$, and $\zeta : \bar{X} \rightarrow \mathbb{R}$ such that, for all $(a, \bar{b}, x) \in A \times \mathcal{B} \times \bar{X}$,*

$$w(a, \bar{b}(a), x) = \xi(x) U(a, \bar{b}(a)) + \zeta(x), \quad \xi(x) > 0. \quad (9)$$

Proof: Let $I, I', I'', I''' \in \mathcal{I}$, $a, a', a'', a''' \in A$ and $\bar{b}(a), \bar{b}(a'), \bar{b}(a''), \bar{b}(a''')$ be as in definition 3. Then, for all $x, x' \in \bar{X}$, $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$, $I_{-x}(a'', \bar{b}(a'')) \sim I'_{-x}(a''', \bar{b}(a'''))$, $I''_{-x'}(a, \bar{b}(a)) \sim I'''_{-x'}(a', \bar{b}(a'))$ and $I''_{-x'}(a'', \bar{b}(a'')) \sim I'''_{-x'}(a''', \bar{b}(a'''))$. By

¹⁴By definition, for all (a, x) and θ , $\beta_{(x, x, a, \theta)} = 1$ and $\alpha_{(x, x, a, \theta)} = 0$.

the representation (7), $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w(a_{I(y)}, b_{I(y)}, y) + w(a, \bar{b}(a), x) = \sum_{y \in \bar{X} - \{x\}} w(a_{I'(y)}, b_{I'(y)}, y) + w(a', \bar{b}(a'), x). \quad (10)$$

Similarly, $I_{-x}(a'', \bar{b}(a'')) \sim I'_{-x}(a''', \bar{b}(a'''))$ implies that

$$\sum_{y \in \bar{X} - \{x\}} w(a_{I''(y)}, b_{I''(y)}, y) + w(a'', \bar{b}(a''), x) = \sum_{y \in \bar{X} - \{x\}} w(a_{I'''(y)}, b_{I'''(y)}, y) + w(a''', \bar{b}(a'''), x), \quad (11)$$

$I''_{-x'}(a, \bar{b}(a)) \sim I'''_{-x'}(a', \bar{b}(a'))$ implies that

$$\sum_{y \in \bar{X} - \{x'\}} w(a_{I''(y)}, b_{I''(y)}, y) + w(a, \bar{b}(a), x') = \sum_{y \in \bar{X} - \{x'\}} w(a_{I'''(y)}, b_{I'''(y)}, y) + w(a', \bar{b}(a'), x'), \quad (12)$$

and $I''_{-x'}(a'', \bar{b}(a'')) \sim I'''_{-x'}(a''', \bar{b}(a'''))$ implies that

$$\sum_{y \in \bar{X} - \{x'\}} w(a_{I''(y)}, b_{I''(y)}, y) + w(a'', \bar{b}(a''), x') = \sum_{y \in \bar{X} - \{x'\}} w(a_{I'''(y)}, b_{I'''(y)}, y) + w(a''', \bar{b}(a'''), x'). \quad (13)$$

But (10) and (11) imply that

$$w(a, \bar{b}(a), x) - w(a', \bar{b}(a'), x) = w(a'', \bar{b}(a''), x) - w(a''', \bar{b}(a'''), x). \quad (14)$$

and (12) and (13) imply that

$$w(a, \bar{b}(a), x') - w(a', \bar{b}(a'), x') = w(a'', \bar{b}(a''), x') - w(a''', \bar{b}(a'''), x'). \quad (15)$$

Define a function $\phi_{(x, x', \bar{b})}$ as follows: $w(\cdot, \bar{b}(\cdot), x) = \phi_{(x, x', \bar{b})} \circ w(\cdot, \bar{b}(\cdot), x')$. Then $\phi_{(x, x', \bar{b})}$ is continuous. Axiom (A.7) implies that $\phi_{(x, x', \bar{b})}$ is monotonic increasing. Moreover, equations (14) and (15) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{(x, x', \bar{b})}$ is affine.

Let $\beta_{(x,o,\bar{b})} > 0$ and $\delta_{(x,o,\bar{b})} := \zeta(x)$ denote, respectively, the multiplicative and additive coefficients corresponding to $\phi_{(x,o,\bar{b})}$, where the inequality follows from the monotonicity of $\phi_{(x,o,\bar{b})}$. Observe that $I_{-o}(a, \bar{b}(a)) \sim I'_{-o}(a', \bar{b}(a'))$ and $I_{-o}(a, \bar{b}'(a)) \sim I'_{-o}(a', \bar{b}'(a'))$ in conjunction with axiom (A.6) imply that

$$\beta_{(x,o,\bar{b})} [w(a, \bar{b}(a), o) - w(a', \bar{b}(a'), o)] = \beta_{(x,o,\bar{b}')} [w(\cdot, \bar{b}'(a), o) - w(a', \bar{b}'(a'), o)] \quad (16)$$

for all $\bar{b}, \bar{b}' \in \mathcal{B}$. Thus, for all $x \in \bar{X}$ and $\bar{b}, \bar{b}' \in \mathcal{B}$, $\beta_{(x,o,\bar{b})} = \beta_{(x,o,\bar{b}')} := \xi(x) > 0$.

Let $a, a' \in A$ and $\bar{b}, \bar{b}' \in \mathcal{B}$ satisfy $(a, \bar{b}(a)) \sim^o (a', \bar{b}'(a'))$. By axiom (A.7) $(a, \bar{b}(a)) \sim^x (a', \bar{b}'(a'))$ if and only if $(a, \bar{b}(a)) \sim^o (a', \bar{b}'(a'))$. By the representation this equivalence implies that

$$w(a, \bar{b}(a), o) = w(a', \bar{b}'(a'), o). \quad (17)$$

if and only if,

$$\xi(x) w(a, \bar{b}(a), o) + \delta_{(x,o,\bar{b})} = \xi(x) w(a', \bar{b}'(a'), o) + \delta_{(x,o,\bar{b}')}. \quad (18)$$

Thus $\delta_{(x,o,\bar{b})} = \delta_{(x,o,\bar{b}')}$. By continuity, (A.2), the conclusion can be extended to \mathcal{B} . Let $\delta_{(x,o,\bar{b})} := \zeta(x)$ for all $\bar{b} \in \mathcal{B}$.

For all $a \in A$ and $\bar{b} \in \mathcal{B}$, define $U(a, \bar{b}(a)) = w(a, \bar{b}(a), o)$ and let $\xi(x)$ and $\zeta(x)$ denote the multiplicative and additive part of $\phi_{(x,o,\bar{b})}$. Then, for all $x \in \bar{X}$,

$$w(a, \bar{b}(a), x) = \xi(x) U(a, \bar{b}(a)) + \zeta(x), \quad \xi(x) > 0. \quad (19)$$

This completes the proof of Lemma 12. ♣

Let $\hat{\alpha}(a, x) = \sum_{\theta \in \Theta} \alpha(a, x, \theta)$, then equations (8) and (9) imply that for every $x \in \bar{X}$, $\bar{b} \in \mathcal{B}$ and $a \in A$,

$$\xi(x) U(a, \bar{b}(a)) + \zeta(x) = H \left(\sum_{\theta \in \Theta} \lambda(a, x, \theta) u(a, \bar{b}(a)(\theta), \theta) + \hat{\alpha}(a, x), a, x \right). \quad (20)$$

Lemma 13 *The identity (20) holds if and only if $u(a, \bar{b}(a)(\theta), \theta) = u(a, \bar{b}(a), \theta')$ for all $\theta, \theta' \in \Theta$, $\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} = \varphi(a)$, $\frac{\hat{\alpha}(a, x)}{\xi(x)} = v(a)$ for all $a \in A$,*

$$\xi(x) [\varphi(a) u(a, \bar{b}(a)) + v(a)] + \zeta(x) = H \left(\sum_{\theta \in \Theta} \lambda(a, x, \theta) u(a, \bar{b}(a)(\theta), \theta) + \hat{\alpha}(a, x), a, x \right), \quad (21)$$

and

$$\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} = U(a, \bar{b}(a)). \quad (22)$$

Proof: (Sufficiency) Let $u(a, \bar{b}(a)(\theta), \theta) := u(a, \bar{b}(a))$ for all $\theta \in \Theta$, $\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} := \varphi(a)$ and $\frac{\hat{\alpha}(a, x)}{\xi(x)} = v(a)$ for all $a \in A$ and suppose that (22) holds. Then equation (20) follows from equation (21).

(Necessity) Multiply and divide the first argument of H by $\xi(x) > 0$. Equation (20) may be written as follows:

$$\xi(x) U(a, \bar{b}(a)) + \zeta(x) = H \left(\xi(x) \left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \right], a, x \right). \quad (23)$$

Define $V(a, \bar{b}(a), x) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)}$ then, for every given $(a, x) \in$

$A \times X$ and all $\bar{b}, \bar{b}' \in \mathcal{B}$,

$$U(a, \bar{b}(a)) - U(a, \bar{b}'(a)) = [H(\xi(x)V(a, \bar{b}(a), x), a, x) - H(\xi(x)V(a, \bar{b}'(a), x), a, x)] / \xi(x). \quad (24)$$

Hence $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$[U(a, \bar{b}(a)) - U(a, \bar{b}'(a))] / [V(a, \bar{b}(a), x) - V(a, \bar{b}'(a), x)] := \kappa(a),$$

is independent of x . Thus

$$\xi(x)U(a, \bar{b}(a)) + \zeta(x) = \kappa(a)\xi(x) \left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \right] + \zeta(x). \quad (25)$$

Hence

$$U(a, \bar{b}(a)) / \kappa(a) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta), \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \quad (26)$$

is independent of x . However, because $\succsim_a^{x'} \neq \succsim_a^x$ for all a and some $x, x' \in \bar{X}$, in general, $\lambda(a, x, \theta)$ is not independent of θ . Moreover, because $\hat{\alpha}(a, x) / \xi(x)$ is independent of \bar{b} , the first term on the right-hand side of (26) must be independent of x . For this to be true $u(a, \bar{b}(a)(\theta), \theta)$ must be independent of θ and $\sum_{\theta \in \Theta} \lambda(a, x, \theta) / \xi(x) := \varphi(a)$ independent of x . Moreover, because the first term on the right-hand side of (26) is independent of x , $\hat{\alpha}(a, x) / \xi(x)$ must also be independent of x . Define $v(a) = \hat{\alpha}(a, x) / \xi(x)$. By definition, \bar{b} is the unique element in its equivalence class that has the property that $u(a, \bar{b}(a)(\theta), \theta)$ is independent of θ . Define $u(a, \bar{b}(a)(\theta), \theta) = u(a, \bar{b}(a))$ for all $\theta \in \Theta$. Hence, $V(a, \bar{b}(a), x)$ is independent of x , thus $V(a, \bar{b}(a), x) = \varphi(a)u(a, \bar{b}(a)) + v(a) = U(\bar{b}(a), a)$ and, consequently, $\kappa(a) = 1$.

Thus

$$U(a, \bar{b}(a)) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} u(a, \bar{b}(a)(\theta); \theta) + \frac{\hat{\alpha}(a, x)}{\xi(x)}. \quad (27)$$

This completes the proof of Lemma 13. ♣

Define $\tilde{u}(a, b(\theta); \theta) = [\varphi(a) u(a, b(\theta); \theta) + v(a)]$ then, by equation (27),

$$U(a, \bar{b}(a)) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x) \varphi(a)} [\varphi(a) u(a, \bar{b}(a)(\theta); \theta) + v(a)] = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\xi(x)} \tilde{u}(a, \bar{b}(a)(\theta); \theta). \quad (28)$$

But, by Lemma 13, $\sum_{\theta \in \Theta} \lambda(a, x, \theta) = \xi(x) \varphi(a)$. Hence the representation (7) implies

$$I \mapsto \sum_{x \in \bar{X}} \left[\sum_{\theta \in \Theta} \frac{\lambda(a_{I(x)}, x, \theta)}{\sum_{\theta \in \Theta} \lambda(a_{I(x)}, x, \theta)} \tilde{u}(a_{I(x)}, \bar{b}(a_{I(x)})(\theta); \theta) \right]. \quad (29)$$

For every $(a, b) \in A \times B$ let $\bar{b} \in \mathcal{B}(\succ)$ be such that $(a, b) \sim^x (a, \bar{b}(a))$ for all $x \in \bar{X}$. Then,

by the inclusivity of \mathcal{B} ,

$$\sum_{\theta \in \Theta} \frac{\lambda(a_{I(x)}, x, \theta)}{\sum_{\theta \in \Theta} \lambda(a_{I(x)}, x, \theta)} \tilde{u}(a_{I(x)}, \bar{b}_{I(x)}(a_{I(x)})(\theta); \theta) = \sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\sum_{\theta \in \Theta} \lambda(a_{I(x)}, x, \theta)} \tilde{u}(a_{I(x)}, b_{I(x)}(\theta); \theta). \quad (30)$$

Thus, by the representation (29),

$$I \mapsto \sum_{x \in \bar{X}} \left[\sum_{\theta \in \Theta} \frac{\lambda(a, x, \theta)}{\sum_{\theta \in \Theta} \lambda(a_{I(x)}, x, \theta)} \tilde{u}(a_{I(x)}, b_{I(x)}(\theta); \theta) \right].$$

For all $x \in X, a \in A$ and $\theta \in \Theta$, define the joint subjective probability distribution on

$\Theta \times \bar{X}$ by

$$\pi(x, \theta | a) = \frac{\lambda(a, x, \theta)}{\sum_{x' \in \bar{X}} \sum_{\theta' \in \Theta} \lambda(a, x', \theta')}. \quad (31)$$

Since $\sum_{\theta \in \Theta} \lambda(a, x, \theta) = \xi(x) \varphi(a)$, for all $x \in \bar{X}$,

$$\sum_{\theta \in \Theta} \pi(x, \theta | a) = \frac{\xi(x) \varphi(a)}{\sum_{x' \in \bar{X}} \xi(x') \varphi(a)} = \frac{\xi(x)}{\sum_{x' \in \bar{X}} \xi(x')}. \quad (32)$$

Define the subjective probability of $x \in \bar{X}$ as follows:

$$\mu(x) = \frac{\xi(x)}{\sum_{x' \in \bar{X}} \xi(x')}. \quad (33)$$

Then the subjective probability of x is given by the marginal distribution on X induced by the joint distributions $\pi(\cdot, \cdot | a)$ on $X \times \Theta$ and is independent of a .

Define the subjective posterior on Θ distribution by

$$\pi(\theta | x, a) = \frac{\pi(x, \theta | a)}{\mu(x)} = \frac{\lambda(a, x, \theta)}{\sum_{\theta \in \Theta} \lambda(a, x, \theta)}, \quad (34)$$

and define the subjective prior on Θ by:

$$\pi(\theta | o, a) = \frac{\lambda(a, o, \theta)}{\sum_{\theta \in \Theta} \lambda(a, o, \theta)}. \quad (35)$$

Substitute in (29) to obtain the representation (3),

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi(\theta | x, a_{I(x)}) \tilde{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) \right]. \quad (36)$$

(Necessity) The necessity of (A.1), (A.2) and (A.3) follows from Wakker (1989) Theorem III.4.1. To see the necessity of (A.4), suppose that $I_{-x}(a, b_{-\theta}r) \succcurlyeq I_{-x}(a, b'_{-\theta}r')$, $I_{-x}(a, b'_{-\theta}r'') \succcurlyeq I_{-x}(a, b_{-\theta}r''')$, and $I_{-x'}(a', b''_{-\theta}r') \succcurlyeq I_{-x'}(a', b'''_{-\theta}r)$. For all $(a, b, x) \in A \times B \times \bar{X}$ let $G(a, b, x) := \sum_{\theta' \in \Theta - \{\theta\}} \pi(\theta' | x, a) \tilde{u}(a, b(\theta'), \theta')$ then, by the representation (3),

$$G(a, b, x) + \pi(\theta | x, a) \tilde{u}(a, r, \theta) \geq G(a, b', x) + \pi(\theta | x, a) \tilde{u}(a, r', \theta) \quad (37)$$

$$G(a, b', x) + \pi(\theta | x, a) \tilde{u}(a, r'', \theta) \geq G(a, b, x) + \pi(\theta | x, a) \tilde{u}(a, r''', \theta) \quad (38)$$

and

$$G(a, b'', x') + \pi(\theta | x', a') \tilde{u}(a', r', \theta) \geq G(a', b''', x') + \pi(\theta | x', a') \tilde{u}(a', r, \theta). \quad (39)$$

But (37) and (38) imply that

$$\tilde{u}(a, r'', \theta) - \tilde{u}(a, r''', \theta) \geq \frac{G(a, b, x) - G(a, b', x)}{\pi(\theta | x, a)} \geq \tilde{u}(a, r', \theta) - \tilde{u}(a, r, \theta). \quad (40)$$

Inequality (39) implies

$$\tilde{u}(a', r', \theta) - \tilde{u}(a', r, \theta) \geq \frac{\sum_{\theta' \in \Theta - \{\theta\}} \pi(\theta' | x', a') [\tilde{u}(a', b'''(\theta'), \theta') - \tilde{u}(a', b''(\theta'), \theta')]}{\pi(\theta | x', a')} \quad (41)$$

But (40) and (41) imply that

$$\tilde{u}(a', r'', \theta) - \tilde{u}(a', r''', \theta) \geq \frac{\sum_{\theta' \in \Theta - \{\theta\}} \pi(\theta' | x', a') [\tilde{u}(a', b'''(\theta'), \theta') - \tilde{u}(a', b''(\theta'), \theta')]}{\pi(\theta | x', a')}. \quad (42)$$

Hence

$$\sum_{\theta' \in \Theta - \{\theta\}} \pi(\theta' | x', a') [\tilde{u}(a', b''(\theta'), \theta') - \tilde{u}(a', b'''(\theta'), \theta')] + \pi(\theta | x', a') [\tilde{u}(a', r'', \theta) - \tilde{u}(a', r''', \theta)] \geq 0. \quad (43)$$

Thus, $I_{-x'}(a', b''_{-\theta} r'') \succcurlyeq I_{-x'}(a', b'''_{-\theta} r''')$.

Let $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, satisfy $I_{-o}(a, b) \sim I_{-o}(a, b')$. Then, by (36),

$$\sum_{\theta \in \Theta} \pi(\theta | o, a) \tilde{u}(a, b(\theta), \theta) = \sum_{\theta \in \Theta} \pi(\theta | o, a) \tilde{u}(a, b'(\theta), \theta) \quad (44)$$

and, by axiom (A.5) and (36)

$$\sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \sum_{\theta \in \Theta} \pi(\theta | x, a) \tilde{u}(a, b(\theta), \theta) = \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \sum_{\theta \in \Theta} \pi(\theta | x, a) \tilde{u}(a, b'(\theta), \theta). \quad (45)$$

Thus

$$\sum_{\theta \in \Theta} [\tilde{u}(a, b(\theta), \theta) - \tilde{u}(a, b'(\theta), \theta)] \left[\pi(\theta | o, a) - \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \pi(\theta | x, a) \right] = 0. \quad (46)$$

This implies that $\pi(\theta | o, a) = \sum_{x \in X} \mu(x) \pi(\theta | x, a) / [1 - \mu(0)]$.

(If $\pi(\theta | o, a) > \sum_{x \in X} \mu(x) \pi(\theta | x, a) / [1 - \mu(0)]$ for some θ and $\mu(o) \pi(\theta' | o, a) < \sum_{x \in X} \mu(x) \pi(\theta' | x, a) / [1 - \mu(0)]$ for some θ' , let $\hat{b}, \hat{b}' \in B$ be such that $\hat{b}(\theta) > b(\theta)$ and $\hat{b}(\hat{\theta}) = b(\hat{\theta})$ for all $\hat{\theta} \in \Theta - \{\theta\}$, $\hat{b}'(\theta') > b'(\theta')$ and $\hat{b}'(\hat{\theta}) = b'(\hat{\theta})$ for all $\hat{\theta} \in \Theta - \{\theta'\}$ and $I_{-o}(a, \hat{b}) \sim I_{-o}(a, \hat{b}')$. Then

$$\sum_{\theta \in \Theta} [\tilde{u}(a, \hat{b}(\theta), \theta) - \tilde{u}(a, \hat{b}'(\theta), \theta)] \left[\pi(\theta | o, a) - \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \pi(\theta | x, a) \right] > 0. \quad (47)$$

But this contradicts (A.5)). This completes the proof of (a).

(b) Suppose, by way of negation, that there exist continuous, real-valued function \hat{u} on $A \times \mathbb{R} \times \Theta$ and, for every $a \in A$, there is a joint probability measure $\hat{\pi}(\cdot, \cdot | a)$ on $\bar{X} \times \Theta$, distinct from those that figure in the representation (3), such that \succsim on \mathcal{I} is represented by

$$I \mapsto \sum_{x \in \bar{X}} \hat{\mu}(x) \left[\sum_{\theta \in \Theta} \hat{\pi}(\theta | x, a_{I(x)}) \hat{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) \right], \quad (48)$$

where $\hat{\mu}(x) = \sum_{\theta \in \Theta} \hat{\pi}(x, \theta | a)$ for all $x \in \bar{X}$, and $\hat{\pi}(\theta | x, a) = \hat{\pi}(x, \theta | a) / \hat{\mu}(x)$ for all $(\theta, x, a) \in \Theta \times X \times A$.

Define $\kappa(x) = \hat{\mu}(x) / \mu(x)$, for all $x \in \bar{X}$. Then the representation (48) may be written

as

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi(\theta | x, a_{I(x)}) \tilde{\gamma}(\theta, x, a_{I(x)}) \kappa(x) \hat{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) \right]. \quad (49)$$

Hence, by (3), $\hat{u}(a, b(\theta), \theta) = u(a, b(\theta), \theta) / \tilde{\gamma}(\theta, x, a) \kappa(x)$. Thus $\tilde{\gamma}(\theta, x, a) \kappa(x)$ is independent of x . Let $\tilde{\gamma}(\theta, x, a) \kappa(x) = \gamma(\theta, a)$. Then, for $\bar{b} \in \mathcal{B}$,

$$I \mapsto \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in \Theta} \pi(\theta | x, a_{I(x)}) \frac{u(a, \bar{b}(a))}{\gamma(\theta, a)} \right]. \quad (50)$$

Let $\hat{b} \in B$ be defined by $u(a, \hat{b}(\theta), \theta) = u(a, \bar{b}(a)) / \gamma(\theta, a)$ for all $\theta \in \Theta$ and $a \in A$. Then $\hat{b} \sim_a^x \bar{b}(a)$ for all $x \in \bar{X}$, and, by definition 3, $\hat{b} \in \mathcal{B}$. Moreover, if $\gamma(\cdot, \cdot)$ is not a constant function then $\hat{b} \neq \bar{b}$. This contradicts the uniqueness of \bar{b} in definition 3. Thus $\gamma(\theta, a) = \bar{\gamma}$ for all $\theta \in \Theta$ and $a \in A$. But

$$1 = \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \hat{\pi}(\theta, x | a_{I(x)}) = \bar{\gamma} \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta, x | a) = \bar{\gamma}. \quad (51)$$

Hence, $\hat{\pi}(\theta, x | a) = \pi(\theta, x | a)$ for all $(\theta, x) \in \Theta \times \bar{X}$ and $a \in A$.

Next consider the uniqueness of the utility functions. Clearly, if $\hat{u}(a, \cdot, \theta) = m u(a, \cdot, \theta) + k$, $m, k > 0$, for all $a \in A$ and $\theta \in \Theta$, then

$$\sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta | x, a) \hat{u}(a, b(\theta), \theta) = m \sum_{x \in \bar{X}} \sum_{\theta \in \Theta} \pi(\theta, x | a) u(a, b(\theta), \theta) + k, \quad (52)$$

and $\{\hat{u}(a, \cdot, \theta) | a \in A, \theta \in \Theta\}$ is another utility function that, jointly with $\{\pi(\cdot, \cdot | a)\}_{a \in A}$ represents \succsim .

Suppose that $\hat{u}(a, \cdot, \theta) = m(a, \theta) u(a, \cdot, \theta) + k$, where $m(\cdot, \cdot)$ is not a constant function. Define $\hat{b}(\theta, a)$ by $m(a, \theta) u(a, \hat{b}(\theta, a), \theta) = u(a, \bar{b}(a))$ for all $\theta \in \Theta$ and $a \in A$. That such

\hat{b} exists follows from the exclusivity of \mathcal{B} . By definition, $\hat{b}(\cdot, a) \sim_a^x \bar{b}(a)$ for all x and $\hat{b} \neq \bar{b}$.

This contradicts the uniqueness of \bar{b} in definition 3. Hence $m(a, \theta)$ must be a constant function.

Consider next $\hat{u}(a, \cdot, \theta) = mu(a, \cdot, \theta) + k(\theta, a)$, and suppose that $k(\cdot, a)$ is not a constant function. Let $\bar{k}(x, a) = \sum_{\theta \in \Theta} \pi(\theta | x, a) k(\theta, a)$. Take $a, a' \in A$ and $\bar{b}, \bar{b}' \in \mathcal{B}$ such that $I_{-x'}(a, \bar{b}(a)) \sim I'_{-x'}(a', \bar{b}(a'))$ and $[\bar{k}(x, a) - \bar{k}(x, a')] \neq 0$ for some strategies I, I' and observation x' . Let

$$J = \sum_{x \in \bar{X} - \{x'\}} \sum_{\theta \in \Theta} [\pi(\theta, x | a_{I(x)}) u(a_{I(x)}, b_{I(x)}(\theta), \theta) - \pi(\theta | x, a_{I'(x)}) u(a_{I'(x)}, b_{I'(x)}(\theta), \theta)]$$

and

$$\hat{J} = \sum_{x \in \bar{X} - \{x'\}} \sum_{\theta \in \Theta} [\pi(\theta, x | a_{I(x)}) \hat{u}(a_{I(x)}, b_{I(x)}(\theta), \theta) - \pi(\theta | x, a_{I'(x)}) \hat{u}(a_{I'(x)}, b_{I'(x)}(\theta), \theta)].$$

Then

$$\hat{u}(a, \bar{b}(a)) - \hat{u}(a', \bar{b}(a')) + \hat{J} = m[u(a, \bar{b}(a)) - u(a', \bar{b}(a')) + J] + [\bar{k}(x, a) - \bar{k}(x, a')]. \quad (53)$$

But $u(a, \bar{b}(a)) - u(a', \bar{b}(a')) + J = 0$ and, by equation (53) $\hat{u}(a, \bar{b}(a)) - \hat{u}(a', \bar{b}(a')) + \hat{J} \neq 0$.

Hence $\hat{u}(\cdot, \theta)$ does not represent \succsim . This completes the proof of (b).

(c) Next I show that if $\bar{b} \in B^A$ satisfies $\tilde{u}(a, \bar{b}(a)(\theta), \theta) = \tilde{u}(a, \bar{b}(a)(\theta'), \theta')$ for all $\theta, \theta' \in \Theta, a \in A$ then $\bar{b} \in \mathcal{B}$. Let $\tilde{u}(a, \bar{b}(a)(\theta), \theta) = g(a, \bar{b}(a))$. Suppose that representation (3) holds and let $I, I', I'', I''' \in \mathcal{I}, a, a', a'', a''' \in A$ and $x, x' \in \bar{X}$, such that $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a')), I'_{-x}(a'', \bar{b}(a'')) \sim I_{-x}(a''', \bar{b}(a'''))$ and $I''_{-x'}(a', \bar{b}(a')) \sim I'''_{-x'}(a, \bar{b}(a))$. Then

the representation (7) implies that

$$\sum_{\hat{x} \in \bar{X} - \{x\}} w(a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}) + \mu(x) g(a, \bar{b}(a)) = \sum_{\hat{x} \in \bar{X} - \{x\}} w(a_{I'(\hat{x})}, b_{I'(\hat{x})}, \hat{x}) + \mu(x) g(a', \bar{b}(a')) \quad (54)$$

$$\sum_{\hat{x} \in \bar{X} - \{x\}} w(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}) + \mu(x) g(a'', \bar{b}(a'')) = \sum_{\hat{x} \in \bar{X} - \{x\}} w(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}) + \mu(x) g(a''', \bar{b}(a''')) \quad (55)$$

and

$$\sum_{\hat{x} \in \bar{X} - \{x'\}} w(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}) + \mu(x') g(a', \bar{b}(a')) = \sum_{\hat{x} \in \bar{X} - \{x'\}} w(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}) + \mu(x') g(a, \bar{b}(a)). \quad (56)$$

Equations (54) and (55) imply that

$$g(a, \bar{b}(a)) - g(a', \bar{b}(a')) = g(a'', \bar{b}(a'')) - g(a''', \bar{b}(a''')). \quad (57)$$

Equality (56) implies

$$\frac{\sum_{\hat{x} \in \bar{X} - \{x'\}} [w(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}) - w(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x})]}{\mu(x')} = g(a, \bar{b}(a)) - g(a', \bar{b}(a')). \quad (58)$$

Thus

$$\sum_{\hat{x} \in \bar{X} - \{x'\}} w(a_{I''(\hat{x})}, b_{I''(\hat{x})}, \hat{x}) + g(a''', \bar{b}(a''')) = \sum_{\hat{x} \in \bar{X} - \{x'\}} w(a_{I'''(\hat{x})}, b_{I'''(\hat{x})}, \hat{x}) + g(a'', \bar{b}(a'')) \quad (59)$$

Hence $I''_{-x'}(a'', \bar{b}(a'')) \sim I'''_{-x'}(a''', \bar{b}(a'''))$ and $\bar{b} \in \mathcal{B}$.

To show the necessity of (A.5) let $a \in A$, $I \in \mathcal{I}$ and $b, b' \in B$, by the representation $I_{-o}(a, b) \sim I_{-o}(a, b')$ if and only if

$$\sum_{\theta \in \Theta} \pi(\theta | o, a) \tilde{u}(a, b(\theta), \theta) = \sum_{\theta \in \Theta} \pi(\theta | o, a) \tilde{u}(a, b'(\theta), \theta). \quad (60)$$

But $\pi(\theta \mid o, a) = \sum_{x \in X} \mu(x) \pi(\theta \mid x, a) / [1 - \mu(0)]$. Thus (60) holds if and only if

$$\sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \tilde{u}(a, b(\theta), \theta) = \sum_{x \in X} \mu(x) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \tilde{u}(a, b'(\theta), \theta). \quad (61)$$

But (61) is valid if and only if $I^{-o}(a, b) \sim I^{-o}(a, b')$.

For all I and x , let $K(I, x) = \sum_{y \in X - \{x\}} \mu(y) \sum_{\theta \in \Theta} \pi(\theta \mid x, a) \tilde{u}(a_{I(y)}, b_{I(y)}(\theta), \theta)$. To show the necessity of (A.6) Then $I_{-x}(a, \bar{b}(a)) \succcurlyeq I'_{-x}(a', \bar{b}(a'))$ if and only if

$$K(I, x) + \tilde{u}(a, \bar{b}(a)) \geq K(I', x) + \tilde{u}(a', \bar{b}(a')) \quad (62)$$

if and only if

$$K(I, x) + \tilde{u}(a, \bar{b}'(a)) \geq K(I', x) + \tilde{u}(a', \bar{b}'(a')) \quad (63)$$

if and only if $I_{-x}(a, \bar{b}'(a)) \succcurlyeq I'_{-x}(a', \bar{b}'(a'))$. ■

4.2 Proof of Theorem 9

Suppose that \succcurlyeq on \mathcal{I} satisfies (A.1)-(A.7), $\mathcal{B}(\succcurlyeq)$ is inclusive and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$. Let $\bar{b}, \bar{b}', \bar{b}'' \in \mathcal{B}$ and $I, I' \in \mathcal{I}$ satisfy the following conditions: (a) $I_{-x}(a, \bar{b}(a)) \sim I'_{-x}(a', \bar{b}(a'))$, (b) $I_{-x}(a, \bar{b}''(a)) \sim I'_{-x}(a', \bar{b}''(a'))$, and (c) $I_{-x}(a, \bar{b}''(a)) \succ I_{-x}(a, \bar{b}(a)) \succ I'_{-x}(a, \bar{b}'(a))$.

Then, by Theorem 5, (a) and (b) imply that

$$\tilde{u}(a, \bar{b}(a)) - \tilde{u}(a, \bar{b}''(a)) = \tilde{u}(a', \bar{b}(a')) - \tilde{u}(a', \bar{b}''(a')). \quad (64)$$

By Definition 8, $(a, E) \sim_L^x (a', E)$ if and only if $I_{-x}(a, \bar{b}^*(a)_E \bar{b}'(a)) \sim I'_{-x}(a', \bar{b}^*(a')_E \bar{b}'(a'))$, where $I_{-x}(a, \bar{b}^*(a)) \sim I'_{-x}(a', \bar{b}^*(a'))$, $I_{-x}(a, \bar{b}'(a)) \sim I'_{-x}(a', \bar{b}'(a'))$ and $I_{-x}(a, \bar{b}^*(a)) \succ I'_{-x}(a, \bar{b}'(a))$, for $\bar{b}^* \in \{\bar{b}, \bar{b}''\}$. For every strategy I and observation x , let

$$k(I_{-x}) = \sum_{x' \in \bar{X} - \{x\}} \sum_{\theta \in \Theta} \pi(x', \theta | a_{I(x)}) \tilde{u}(a_{I(x')}, b_{I(x')}(\theta), \theta).$$

By Theorem 7, and conditions (a) and (b),

$$I_{-x}(a, \bar{b}(a)_E \bar{b}'(a)) \sim I'_{-x}(a', \bar{b}(a')_E \bar{b}'(a')) \quad (65)$$

if and only if

$$k(I_{-x}) + \eta_a(E) \tilde{u}(a, \bar{b}(a)) + (1 - \eta_a(E)) \tilde{u}(a, \bar{b}'(a)) = \quad (66)$$

$$k(I'_{-x}) + \eta_{a'}(E) \tilde{u}(a', \bar{b}(a')) + (1 - \eta_{a'}(E)) \tilde{u}(a', \bar{b}'(a'))$$

and

$$I_{-x}(a, \bar{b}''(a)_E \bar{b}'(a)) \sim I'_{-x}(a', \bar{b}''(a')_E \bar{b}'(a')) \quad (67)$$

if and only if

$$k(I_{-x}) + \eta_a(E) \tilde{u}(a, \bar{b}''(a)) + (1 - \eta_a(E)) \tilde{u}(a, \bar{b}'(a)) = \quad (68)$$

$$k(I'_{-x}) + \eta_{a'}(E) \tilde{u}(a', \bar{b}''(a')) + (1 - \eta_{a'}(E)) \tilde{u}(a', \bar{b}'(a')).$$

Hence

$$\eta_a(E) [\tilde{u}(a, \bar{b}(a)) - \tilde{u}(a, \bar{b}''(a))] = \eta_{a'}(E) [\tilde{u}(a', \bar{b}(a')) - \tilde{u}(a', \bar{b}''(a'))]. \quad (69)$$

Equation (64) implies that equation (69) holds if and only if $\eta_a(E) = \eta_{a'}(E)$. Thus $(a, E) \sim_L^x (a', E)$ for all $a, a' \in A$ and $E \in \mathcal{E}_a \cap \mathcal{E}_{a'}$ if and only if $\eta_a(E) = \eta_{a'}(E)$. Hence \succ on \mathcal{I} satisfies (A.9) if and only if $\eta_a(E) = \eta_{a'}(E)$. ■

4.3 Proof of Theorem 11

Let \mathcal{A} denote the algebra generated by $(\bigwedge_{i=1}^k \mathcal{T}_{a_i}) \wedge \mathcal{Y}$, the join of the partitions $\mathcal{T}_{a_1}, \dots, \mathcal{T}_{a_k}$, where k is an integer and \mathcal{Y} . Elements of \mathcal{A} include all the intersections of sets belonging to distinct partitions $\mathcal{T}_{a_1}, \dots, \mathcal{T}_{a_k}$. Because $(S, A, \Theta, \bar{X}, t)$ is rich each event involving such interactions is equal to a “simple” event, $E \in \mathcal{E}_a$, for some $a \in A$. By Theorem 7, the probability of E is unique and is given by $\eta_a(E)$. Define the probability measure P_0 on \mathcal{A} as follows:

$$P_0(E) = \eta_a(E), \quad (70)$$

for all $E \in \mathcal{A}$.

But \mathcal{E} is the σ -algebra generated by \mathcal{A} . Hence, by Billingsley (1986) Theorem 3.1, P_0 has a unique extension, P , to \mathcal{E} . Then P is a probability measure on (S, \mathcal{E}) . In particular,

$$P(\{x\} \times t^{-1}(\theta, a)) = \eta_a(\{x\} \times t^{-1}(\theta, a)) = \pi(x, \theta | a) \quad (71)$$

and

$$P(\{x\} \times \Omega) = \sum_{\theta \in \Theta} \pi(x, \theta | a) = \mu(x). \quad (72)$$

Substitute $P(\{x\} \times t^{-1}(\theta, a))$ for $\pi(\theta | x, a)$ and $P(\{x\} \times \Omega)$ for $\mu(x)$ in Theorem 5, to obtain the representation (6). ■

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